Constant False Alarm Rate (CFAR)

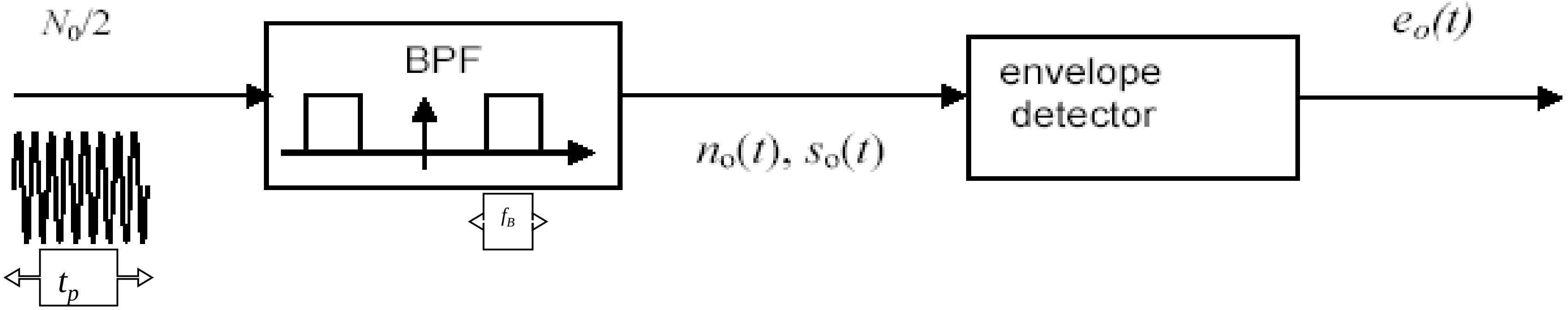
or

Detection with adaptive threshold

1) 1-parameter background distribution
2) Single pulse (no integration)

Fixed threshold non-coherent detection - Review

Detecting a single pulse of amplitude A, pulse-width t_p and frequency ω_c, in the presence of AWGN with power spectral density $N_0/2$, and input BPF bandwidth of f_B.



The noise at the output of the BPF has RMS β.

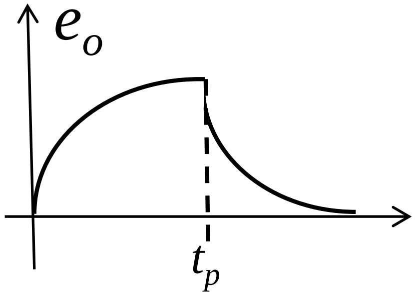
$$n_o(t) = X(t)\cos(\omega_c t) + Y(t)\sin(\omega_c t)$$

$$\overline{X^2(t)} = \overline{Y^2(t)} = \beta^2 \, , \; \beta = \left[\overline{n_o^2(t)} \right]^{1/2} = (N_0 f_B)^{1/2}$$

$$X \sim \mathrm{N}(0, \beta^2), \, Y \sim \mathrm{N}(0, \beta^2)$$

f_B is wide enough to allow the output pulse (without noise) to build up to the full amplitude A of the input pulse.

$$e_o(t_p) = r$$



r is the random variable whose value is compared to a threshold, to decide about detection.

We will study r's PDF when A is fixed (non-fluctuating target) or when A is itself a random variable (fluctuating target).

Non-fluctuating target (A is a constant)

$$p(r)=\frac{r}{\beta^2}\exp\frac{-\left(r^2+A^2\right)}{2\beta^2}\mathrm{I}_0\left(\frac{rA}{\beta^2}\right)$$

$$p(r)\Big|_{A=0}=\frac{r}{\beta^2}\exp\frac{-r^2}{2\beta^2},\quad \text{noise only}$$

Rayleigh fluctuating target (A is a r.v.)

$$p(A)=\frac{A}{A_0^2}\exp\frac{-A^2}{2A_0^2}\ ,\ A>0$$

$$z\triangleq\frac{r^2}{2\beta^2}$$

Square-law envelope detection with normalization that assumes knowledge of β

$$p(z|A)=\exp\left[-\left(z+\frac{A^2}{2\beta^2}\right)\right]\mathrm{I}_0\left(\sqrt{\frac{2zA^2}{\beta^2}}\right)$$

$$p(z)=\int_0^\infty p(z|A)\,p(A)\,dA$$

$$\text{hint: }\int_0^\infty \exp(-ax)I_0\left(b\sqrt{x}\right)dx=\frac{1}{a}\exp\left(\frac{b^2}{4a}\right)$$

$$p(z)=D\exp(-Dz)\ ,\ z>0,\ D=\frac{1}{1+\dfrac{A_0^2}{\beta^2}}$$

$$z \triangleq r^2$$

Square-law envelope detection without normalization

$$p(z) = D\exp(-Dz) \quad , \; z > 0$$

$$D = \frac{1}{2\beta^2\left(1+\frac{A_0^2}{\beta^2}\right)}$$

$$D\,2\beta^2 = \frac{1}{1+\frac{A_0^2}{\beta^2}}$$

P_D Probability of detection (namely that z will cross a threshold T)

$$P_D = \int_T^\infty p(z)\,dz = \int_T^\infty D\exp(-Dz)\,dz = \left(e^{-T}\right)^D$$

P_{FA} Probability of false alarm (namely that z will cross a threshold T when A_0=0)

$$A_0 = 0 \Rightarrow D_{FA} = \frac{1}{2\beta^2} \Rightarrow P_{FA} = \left(e^{-T}\right)^{D_{FA}}$$

$$P_{FA} = \left(e^{-T}\right)^{1/2\beta^2}$$

$$e^{-T} = \left(P_{FA}\right)^{2\beta^2}$$

$$P_D = P_{FA}^{\frac{1}{1+\frac{A_0^2}{\beta^2}}}$$

3-way relationship between P_D, P_{FA} and SNR

The strong dependence of P_{FA} on β - numerical example:

We assume $\beta=\beta_0$ and want $P_{FA}=10^{-5}$

Setting the threshold T :

$$e^{-T}=\left(P_{FA}\right)^{2\beta^2} \Rightarrow e^{-T}=\left(10^{-5}\right)^{2\beta_0^2}$$

But in fact $\beta=2\beta_0$. Hence the actual false alarm will be:

$$P_{FA}=\left(e^{-T}\right)^{1/2\beta^2}$$

$$P_{FA}=\left[\left(10^{-5}\right)^{2\beta_0^2}\right]^{1/2(2\beta_0)^2}$$

$$P_{FA}=\left(10^{-5}\right)^{\frac{2\beta_0^2}{8\beta_0^2}}=10^{-5/4}=10^{-1.25}$$

Doubling the noise RMS raised the P_{FA} from 0.00001 to 0.056. Unacceptable !

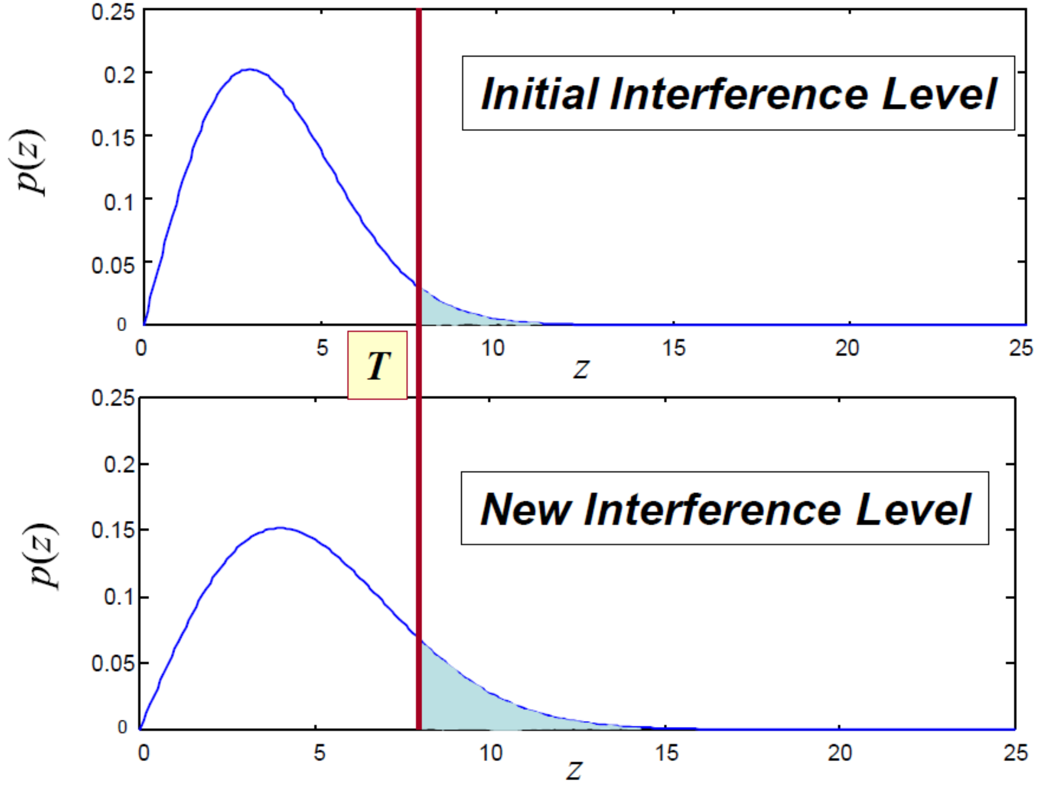
We need an adaptive threshold that will ensure **C**onstant **F**alse **A**larm **R**ate (CFAR).

Justification for CFAR

- The problem is that the interference power isn't known and may change with time, range, or Doppler
  - so we can't accurately compute the threshold required to achieve a given P_{FA}

$\beta =$ noise RMS

$$\beta = \beta_0 \Rightarrow \quad P_{FA} = 0.00001$$
$$\beta = 2\beta_0 \Rightarrow P_{FA} = 0.056$$



From M.A. Richards, Georgia Tech

Basic CFAR Approach

- Solution: *Estimate* the unknown interference parameter(s) from the data
  - So CFAR will be an *adaptive* processor
- Statistics of interference in neighboring range and/or Doppler cells is assumed to be representative of the interference in the cell being tested

Constant False Alarm Rate (CFAR)

- The goal is to adaptively estimate a detection threshold for each cell while maintaining a constant false alarm rate
- To implement a CFAR processor the following are required:
  - a desired false alarm probability
  - an assumed probability density function for the interference (*e.g.*, Rayleigh, Weibull, log-normal)
  - an estimate of the local (cell under test) interference distribution parameters

From M.A. Richards, Georgia Tech

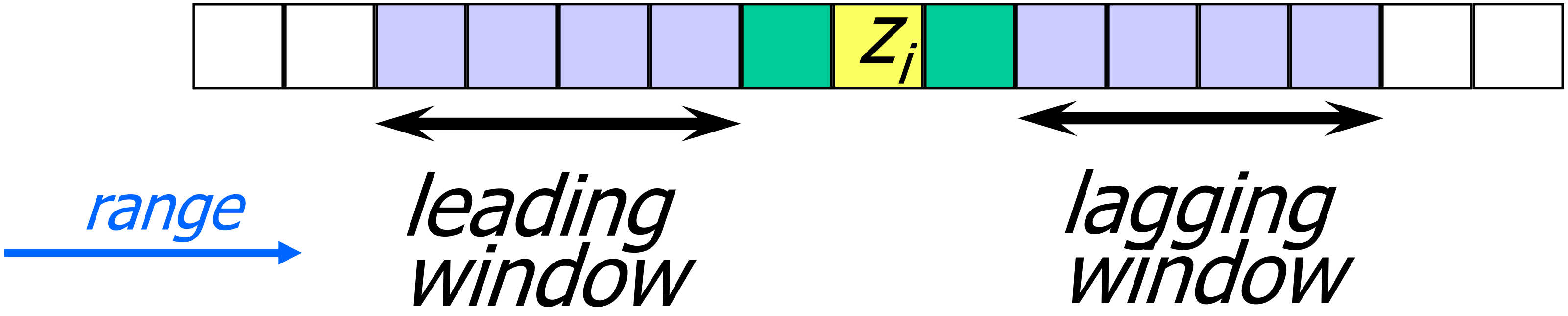
Cell Averaging CFAR (CA-CFAR)

- Threshold set using estimate of the mean of the interference in a reference window
- Optimal estimator for homogeneous interference
  - unbiased
  - minimum variance
- CFAR Loss
  - decreases with increasing reference window size
  - increases with decreasing P_{FA}
- Vulnerable to masking degradation
  - strong interferer in reference window “captures” threshold
  - exclusion of test cell and adjoining cells from the reference window is desired to suppress self-masking.

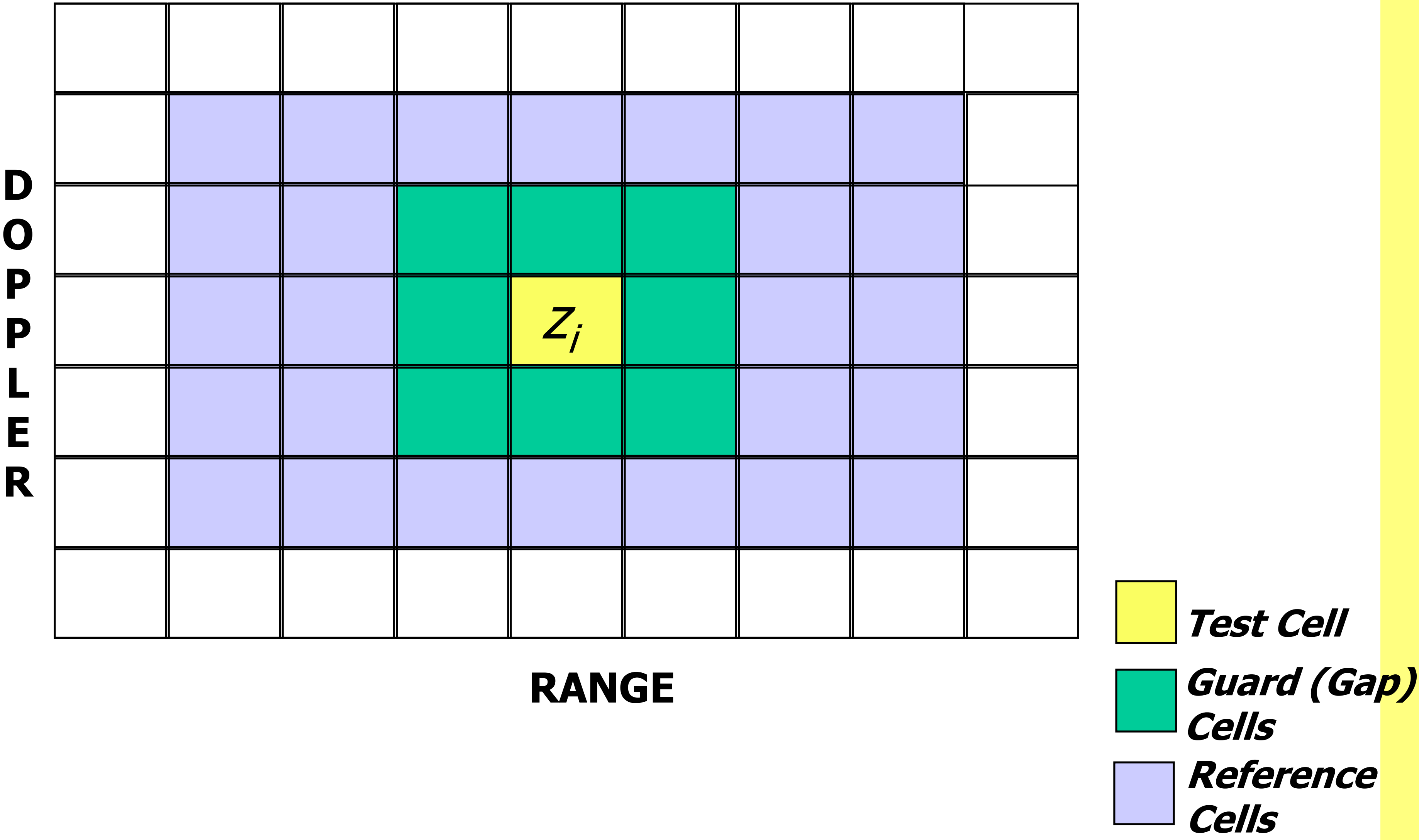
From M.A. Richards, Georgia Tech

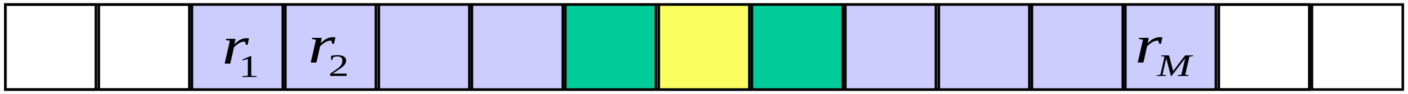
1D CFAR Window Structure

- 🟨 *Test cell*: the value to be compared to the threshold
- 🟩 *Guard* or *gap cell*: value *not* to be included in the interference estimate due to possible target contamination
- 🟪 *Reference cell*: values assumed to be interference only, thus used to estimate interference parameters



2D CFAR Window Structure





M reference background cells. If the reference cells exhibit the envelope r of narrow band noise with RMS β then they are i.i.d. with Rayleigh PDF (from early slide)

$$p(r) = \frac{r}{\beta^2} \exp\left(\frac{-r^2}{2\beta^2} \right), \quad r > 0$$

$$r_0 = \beta \ , \quad \overline{r^2} = 2r_0^2 = 2\beta^2$$

Maximum likelihood estimation of β

$$p(r_1, r_2, ..., r_M) = p(r_1)\, p(r_2) ... p(r_M) = \frac{1}{\beta^{2M}} (r_1 \ r_2 \ ... \ r_M) \exp\left[\frac{-1}{2\beta^2} \left(r_1^2 + r_2^2 + ... + r_M^2 \right) \right]$$

$$\log\left[p(r_1, r_2, ..., r_M) \right] = -2M \ln \beta + \sum_{k=1}^{M} \ \ln r_k - \frac{1}{2\beta^2} \sum_{k=1}^{M} \ r_k^2$$

$$\frac{d}{d\beta} \left\{ \log\left[p(r_1, r_2, ..., r_M) \right] \right\} = \frac{-2M}{\beta} + \frac{1}{\beta^3} \sum_{k=1}^{M} \ r_k^2$$

$$\frac{-2M}{\hat{\beta}} + \frac{1}{\hat{\beta}^3} \sum_{k=1}^{M} \ r_k^2 = 0$$

$$\hat{\beta}^2 = \frac{1}{2M} \sum_{k=1}^{M} \ r_k^2$$

$$\hat{\beta} = \sqrt{\frac{1}{2M} \sum_{k=1}^{M} \ r_k^2}$$

Namely, for a ML estimation of β or β^2 we need $\sum_{k=1}^{M} r_k^2$

which can be obtained from a **square-law** detector.

$z \triangleq r^2$ ← Square-law envelope detection without normalization

$$p(z) = D\exp(-Dz) \quad , \; z > 0$$

$$D = \frac{1}{2\beta^2\left(1 + \frac{A_0^2}{\beta^2}\right)}$$

Assume the reference cells include noise only (no target), i.e., A_0=0

Then at the output of the square-law detector is $z = r^2$

and z is exponentially distributed

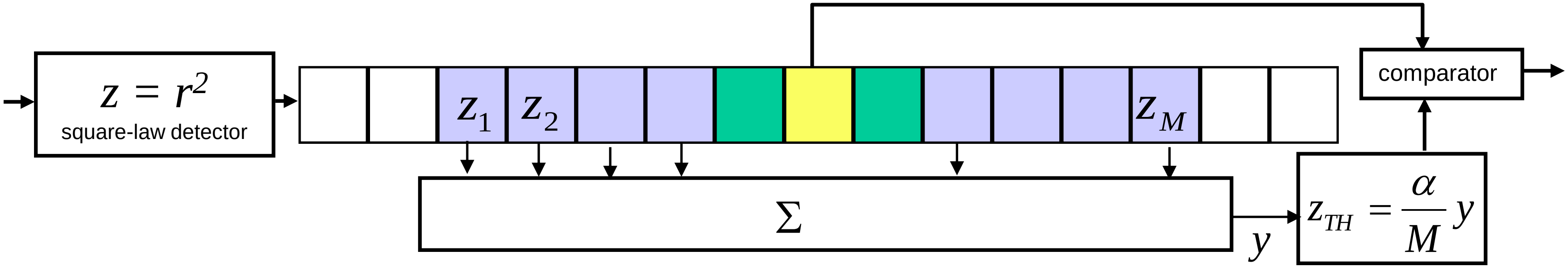
$$p_z(z) = \frac{1}{2\beta^2}\exp\left(\frac{-z}{2\beta^2}\right)$$

$\overline{r^2} = 2\beta^2$

$\bar{z} = 2\beta^2$

$$p_z(z) = \frac{1}{\bar{z}}\exp\left(\frac{-z}{\bar{z}}\right)$$

In CA-CFAR



The sum of M reference cells is $y = \sum_{k=1}^{M} z_k$ The **threshold** is $z_{TH} = \frac{\alpha}{M} y$

$$\frac{y}{M} = \frac{1}{M} \sum_{k=1}^{M} z_k = \bar{Z} \quad \Rightarrow \quad z_{TH} = \alpha \bar{Z}$$

$$z \triangleq r^2 = X^2 + Y^2 \qquad X \sim \mathrm{N}\left(0, \beta^2\right), \, Y \sim \mathrm{N}\left(0, \beta^2\right)$$

$$y = \sum_{k=1}^{2M} X_k^2 \qquad X \sim \mathrm{N}\left(0, \beta^2\right)$$

The PDF of y (ver. 1)

$$y = \sum_{k=1}^{2M} X_k^2$$

$$X \sim \mathrm{N}\left(0, \beta^2\right)$$

$$y = s\beta^2$$

The relation to the Chi-squared distribution

$$s = \sum_{k=1}^{2M} \left(\frac{X_k}{\beta} \right)^2 \quad , \quad \frac{X_k}{\beta} \sim \mathrm{N}\left(0,1\right)$$

s is Chi-squared distributed with $2M$ degrees of freedom

$$p\left(s\right) = \frac{s^{M-1}}{2^M \left(M-1\right)!} \exp\left(\frac{-s}{2} \right), \; s \geq 0 \; , \; \text{zero elsewhere}$$

Perform change of variables (scaling) from s to y , when: $y = s\beta^2$

$$p\left(y\right) = \frac{y^{M-1}}{\left(2\beta^2\right)^M \left(M-1\right)!} \exp\left(\frac{-y}{2\beta^2} \right), \; y \geq 0 \; , \; \text{zero elsewhere}$$

$$y=\sum_{k=1}^{M} z_k$$

The PDF of y (ver. 2)

$$p_y(y)=p_z(z_1)\otimes p_z(z_2)\otimes\ldots\otimes p_z(z_M)$$

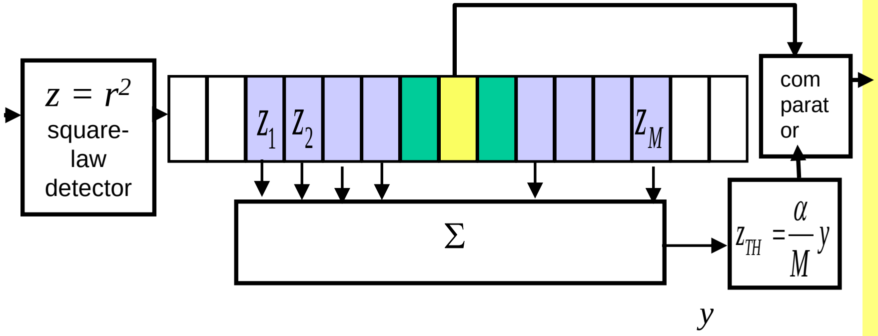
$$p_z(z)=\frac{1}{2\beta^2}\exp\left(\frac{-z}{2\beta^2}\right)$$

Laplace transform of $p_z(z)$

$$h(s)=\mathrm{L}[p_z(z)]$$

$$h(s)=\frac{1}{2\beta^2}\int_0^{\infty}\exp\left[-\left(s+\frac{1}{2\beta^2}\right)z\right]dz=\frac{1}{2\beta^2\left(s+\frac{1}{2\beta^2}\right)}$$

$$\mathrm{L}[p_y(y)]=h^M(s)=\frac{1}{\left(2\beta^2\right)^M}\left(\frac{1}{s+\frac{1}{2\beta^2}}\right)^M$$



$$p_y(y)=\mathrm{L}^{-1}\left[h^M(s)\right]$$

$$\boxed{p_y(y)=\frac{y^{M-1}}{\left(2\beta^2\right)^M(M-1)!}\exp\left(\frac{-y}{2\beta^2}\right),\quad y\geq 0,\ \text{zero elsewhere}}$$

Probability of detection P_D in the cell under test (CUT)

Swerling I or II target with amplitude PDF

$$p_A(A) = \frac{A}{A_0^2}\exp\left(\frac{-A}{A_0^2}\right)$$

From the Detection lectures we have that at the output of square-law detector, when both target and background are Rayleigh distributed:

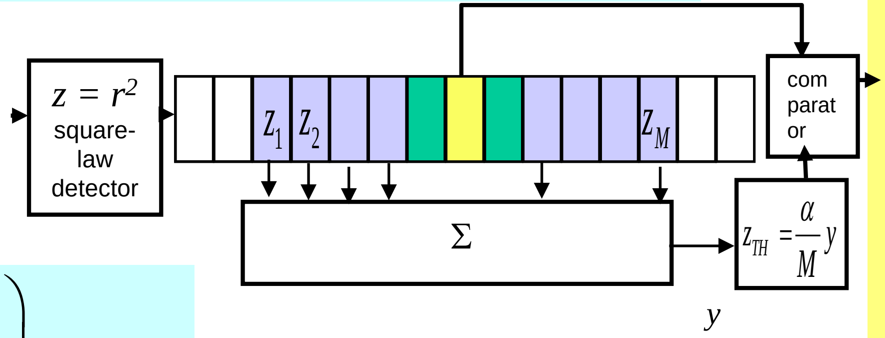
$$p(z_{cut}) = D\exp(-Dz_{cut}) \;\; , \quad D = \frac{1}{2\beta^2\left(1+\dfrac{A_0^2}{\beta^2}\right)}$$

The conditional probability of detection for a given threshold

$$P_D\left(\frac{A_0^2}{\beta^2}\,\Big|\, z_{TH}\right) = \int_{z_{TH}}^{\infty} p(z_{cut})dz_{cut} = D\int_{z_{TH}}^{\infty} \exp(-Dz_{cut})dz_{cut} = \exp(-Dz_{TH})$$

The threshold is

$$z_{TH} = \frac{\alpha}{M}y$$



The unconditional probability of detection is:

$$P_D\left(\frac{A_0^2}{\beta^2},\alpha\right) = \int_{y=0}^{\infty} P_D\left(\frac{A_0^2}{\beta^2}\bigg|_{\left(z_{TH}=\frac{\alpha y}{M}\right)}\right)p(y)dy$$

$$P_D\left(\frac{A_0^2}{\beta^2},\alpha\right)=\int_{y=0}^{\infty} P_D\left(\frac{A_0^2}{\beta^2}\bigg|_{\left(z_{TH}=\frac{\alpha y}{M}\right)}\right)p(y)dy$$

$$P_D\left(\frac{A_0^2}{\beta^2}\,\Big|\,z_{TH}\right)=\exp\left(-Dz_{TH}\right)$$

$$P_D\left(\frac{A_0^2}{\beta^2},\alpha\right)=\int_{y=0}^{\infty}\exp\left(\underbrace{\frac{-1}{2\beta^2\left(1+\frac{A_0^2}{\beta^2}\right)}}\frac{\alpha y}{M}\right)\overbrace{\frac{y^{M-1}}{\left(2\beta^2\right)^M(M-1)!}\exp\left(\frac{-y}{2\beta^2}\right)}dy$$

$$D=\frac{1}{2\beta^2\left(1+\frac{A_0^2}{\beta^2}\right)}$$

$$P_D\left(\frac{A_0^2}{\beta^2},\alpha\right)=\frac{1}{\left(2\beta^2\right)^M(M-1)!}\underbrace{\int_{y=0}^{\infty}y^{M-1}\exp(-sy)dy}\quad,\quad s=\frac{1}{2\beta^2}\left[1+\frac{\alpha}{M\left(1+\frac{A_0^2}{\beta^2}\right)}\right]$$

$$,\ \overbrace{\int_{y=0}^{\infty}y^{M-1}\exp(-sy)dy}=\mathrm{L}\left[y^{M-1}\right]=\frac{(M-1)!}{s^M}$$

$$\frac{A_0^2}{\beta^2}=\overline{SNR}$$

$$P_D\left(\frac{A_0^2}{\beta^2},\alpha\right)=\left[1+\frac{\alpha}{M\left(1+\frac{A_0^2}{\beta^2}\right)}\right]^{-M}$$

Probability of detection P_D in the cell under test (CUT)

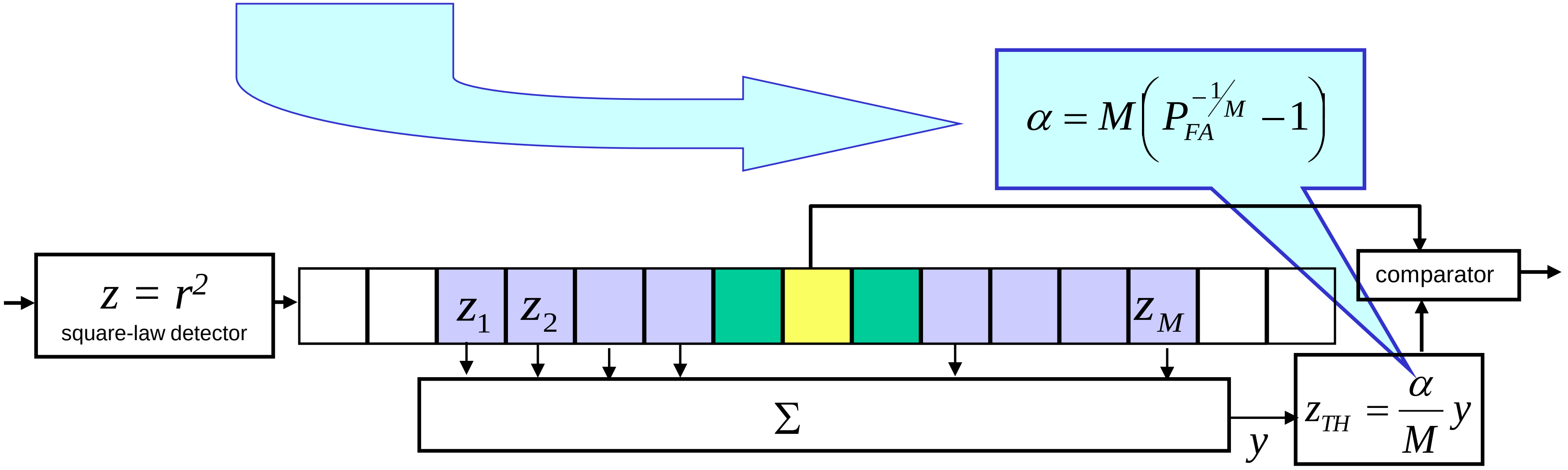
$$\boxed{P_D\left(\overline{SNR},\alpha\right)=\left[1+\frac{\alpha}{M\left(1+\overline{SNR}\right)}\right]^{-M}}$$

$$P_{FA}=P_D\Big|_{\overline{SNR}=0}$$

$$\boxed{P_{FA}(\alpha)=\left(1+\frac{\alpha}{M}\right)^{-M}}$$

← Not a function of β, hence CFAR

← Independent of target PDF, since SNR=0



$$\alpha=M\left(P_{FA}^{-1/M}-1\right)$$

CFAR Threshold

- Given a desired P_{FA} and a known or assumed interference pdf
  - Estimate the local interference parameters
    - *one parameter CFAR*: estimate the local mean
    - *two parameter CFAR*: estimate the local mean and variance
  - Scale the estimate to obtain the desired threshold (based on a given P_{FA} and pdf)

CFAR LOSS

$M \Rightarrow \infty$

➔ the estimate of β is perfect (non-CFAR case)

$$P_{FA}(\alpha) = \left(1 + \frac{\alpha}{M}\right)^{-M}$$

$$\ln P_{FA} = -M \ln\left(1 + \frac{\alpha}{M}\right) = -M\left(\frac{\alpha}{M} - \frac{\alpha^2}{2M^2} + \frac{\alpha^3}{3M^3} - \ldots\right) = -\alpha + \frac{\alpha^2}{2M} - \frac{\alpha^3}{3M^2} - \ldots$$

$$\alpha\Big|_{M\to\infty} = -\ln P_{FA}$$

➔

$$P_{FA}\Big|_{M\to\infty} = \exp(-\alpha)$$

$$P_D\left(\overline{SNR}, \alpha\right) = \left[1 + \frac{\alpha}{M\left(1 + \overline{SNR}\right)}\right]^{-M}$$

$$P_D\Big|_{M\to\infty} = \exp\left(\frac{-\alpha}{1 + \overline{SNR}}\right)$$

$$P_D\Big|_{M\to\infty} = \exp\left(\frac{\ln P_{FA}}{1 + \overline{SNR}}\right) = \left(P_{FA}\right)^{\frac{1}{1+\overline{SNR}}}$$

indeed like in non-CFAR detection

$$\overline{SNR}\Big|_{M\to\infty} = \frac{\log \dfrac{P_{FA}}{P_D}}{\log P_D}$$

For Rayleigh background (and noise) and for Rayleigh target

$$P_{FA}(\alpha)=\left(1+\frac{\alpha}{M}\right)^{-M}$$

$$P_D\left(\overline{SNR},\alpha\right)=\left[1+\frac{\alpha}{M\left(1+\overline{SNR}\right)}\right]^{-M}$$

$$P_D\left(\overline{SNR},\alpha\right)=P_{FA}\left(\alpha_D\right)$$

$$\alpha_D=\left(\frac{\alpha}{1+\overline{SNR}}\right)$$

$$M \ll \infty$$

$$P_{FA}(\alpha) = \left(1+\frac{\alpha}{M}\right)^{-M}$$

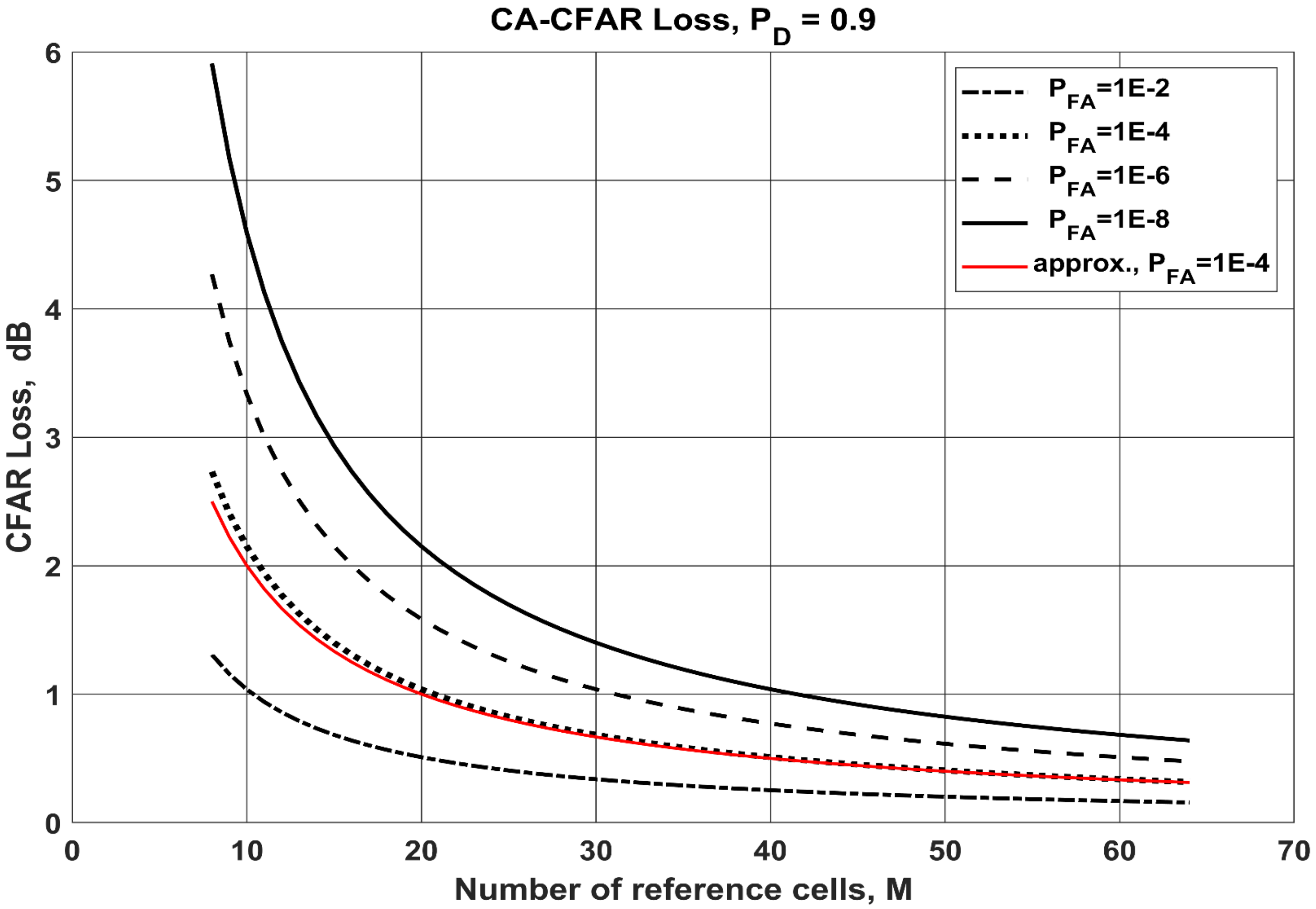
$$\alpha\Big|_{M\ll\infty} = M\left(P_{FA}^{-1/M} - 1\right)$$

$$P_D\left(\overline{SNR},\alpha\right) = \left[1+\frac{\alpha}{M\left(1+\overline{SNR}\right)}\right]^{-M}$$

$$\overline{SNR}\Big|_{M\ll\infty} = \frac{\alpha}{M\left(P_D^{-1/M}-1\right)} - 1 = \frac{M\left(P_{FA}^{-1/M}-1\right)}{M\left(P_D^{-1/M}-1\right)} - 1 = \underbrace{\frac{\left(\frac{P_D}{P_{FA}}\right)^{1/M}-1}{1-P_D^{1/M}}}$$

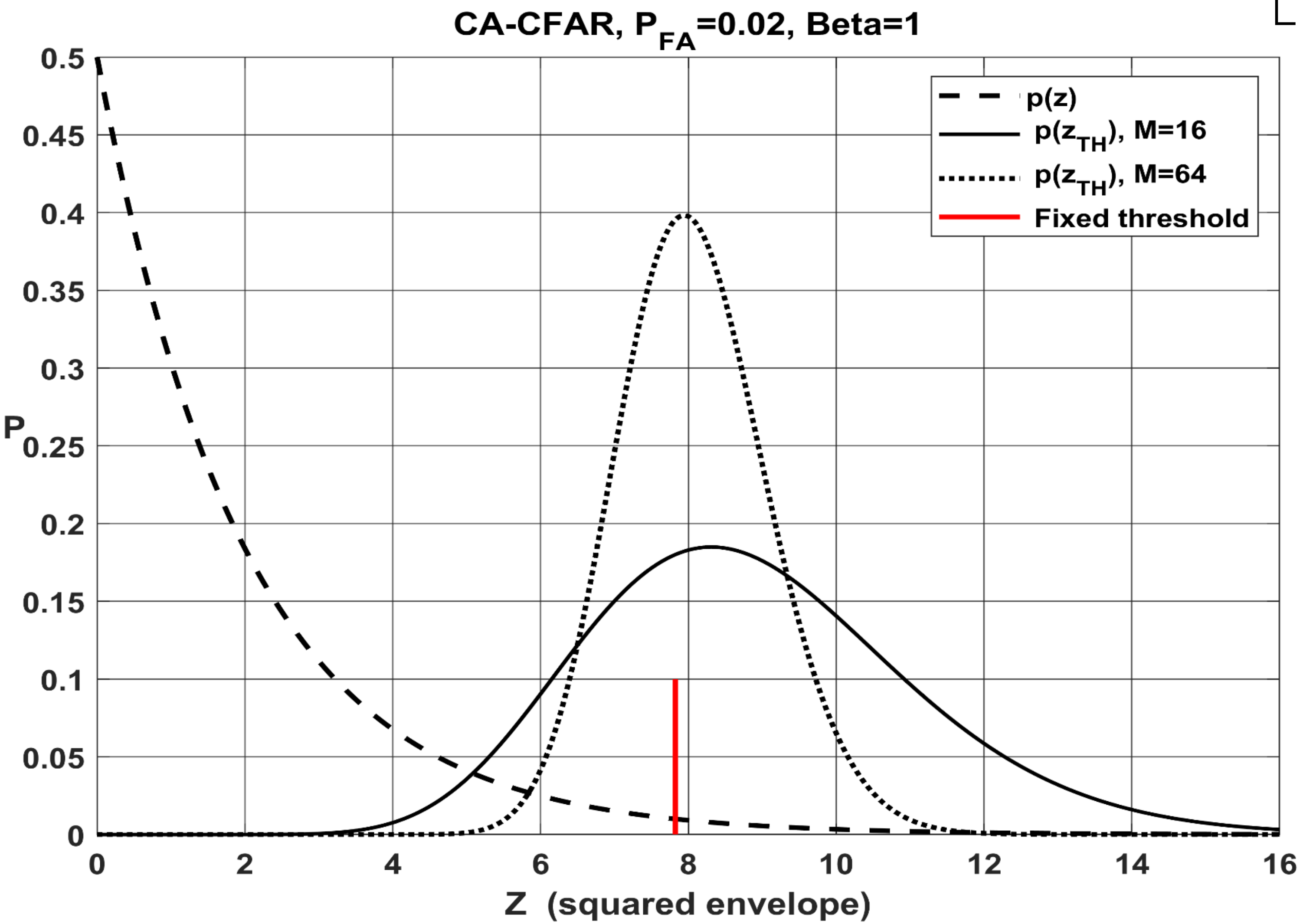
$$\overline{SNR}\Big|_{M\to\infty} = \underbrace{\frac{\log\frac{P_{FA}}{P_D}}{\log P_D}}$$

$$\boxed{\text{CFAR LOSS} = \frac{\overline{SNR}\Big|_{M\ll\infty}}{\overline{SNR}\Big|_{M\to\infty}} \underset{M\geq 16}{\approx} \left(P_{FA}\right)^{\frac{-1}{2M}}}$$



CA-CFAR loss, Swerling 1 or 2

$$\beta = \left[\overline{n_o^2(t)} \right]^{1/2}$$



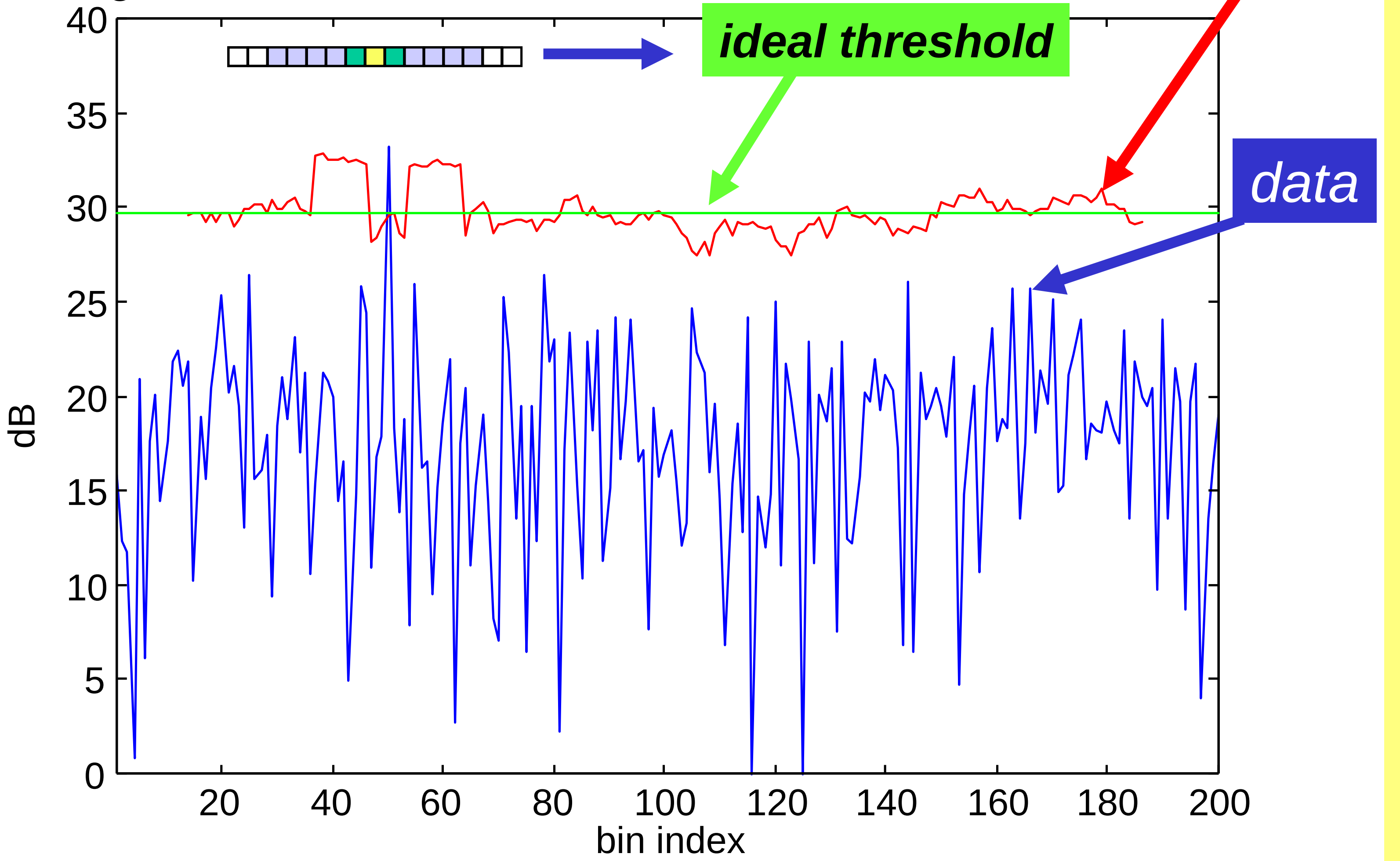
Thresholds PDF for different number of reference cells

CFAR Example Parameters

- The following CFAR example plot is based on:
  - exponentially distributed interference (mean = 20 dB)
    - implies square law detector
  - $P_{FA} = 10^{-4}$
  - Non-fluctuating target, 13 dB above the mean interference

CA-CFAR Example

- *lag/lead window size = 10 cells, gap size = 3 cells*
- *target in range bin #50*



Measures of Effectiveness

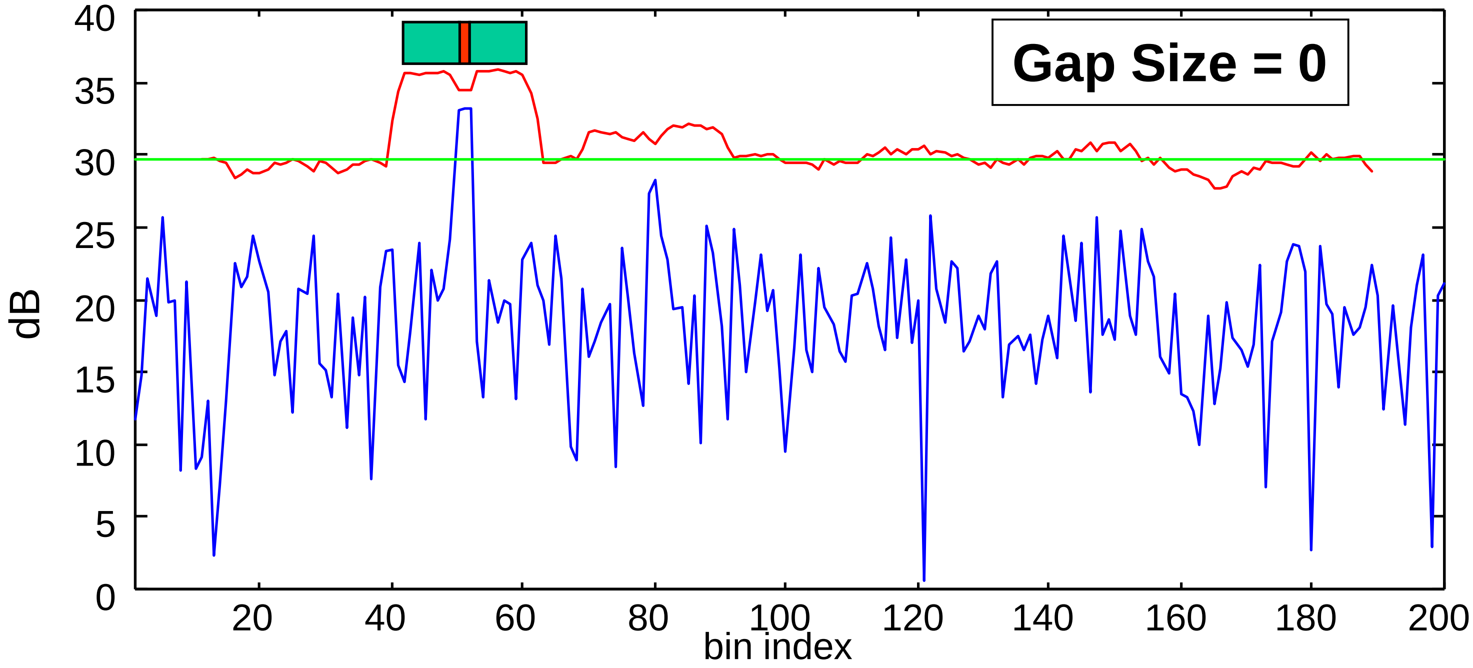
- CFAR Loss
  - Additional SNR required to obtain the corresponding fixed threshold detection performance
  - Is a function of the desired P_{FA}, the interference pdf, and the target model
- Masking (‘Heterogeneous CFAR Loss’)
  - Detection threshold bias due to heterogeneous interference within the reference window.
    - Target masking
    - Clutter-edge masking
- Clutter-Edge False Alarm Suppression
  - False alarms associated with the test cell in the vicinity of a clutter edge boundary

Real-World Problems with CA-CFAR

- In the real world
  - targets can extend over more than one cell, thus getting into the assumed “interference only” cells and distorting the interference estimate
  - multiple closely-spaced targets can distort the interference estimate for each other
  - clutter interference can be *nonhomogeneous* due to changes in physical terrain

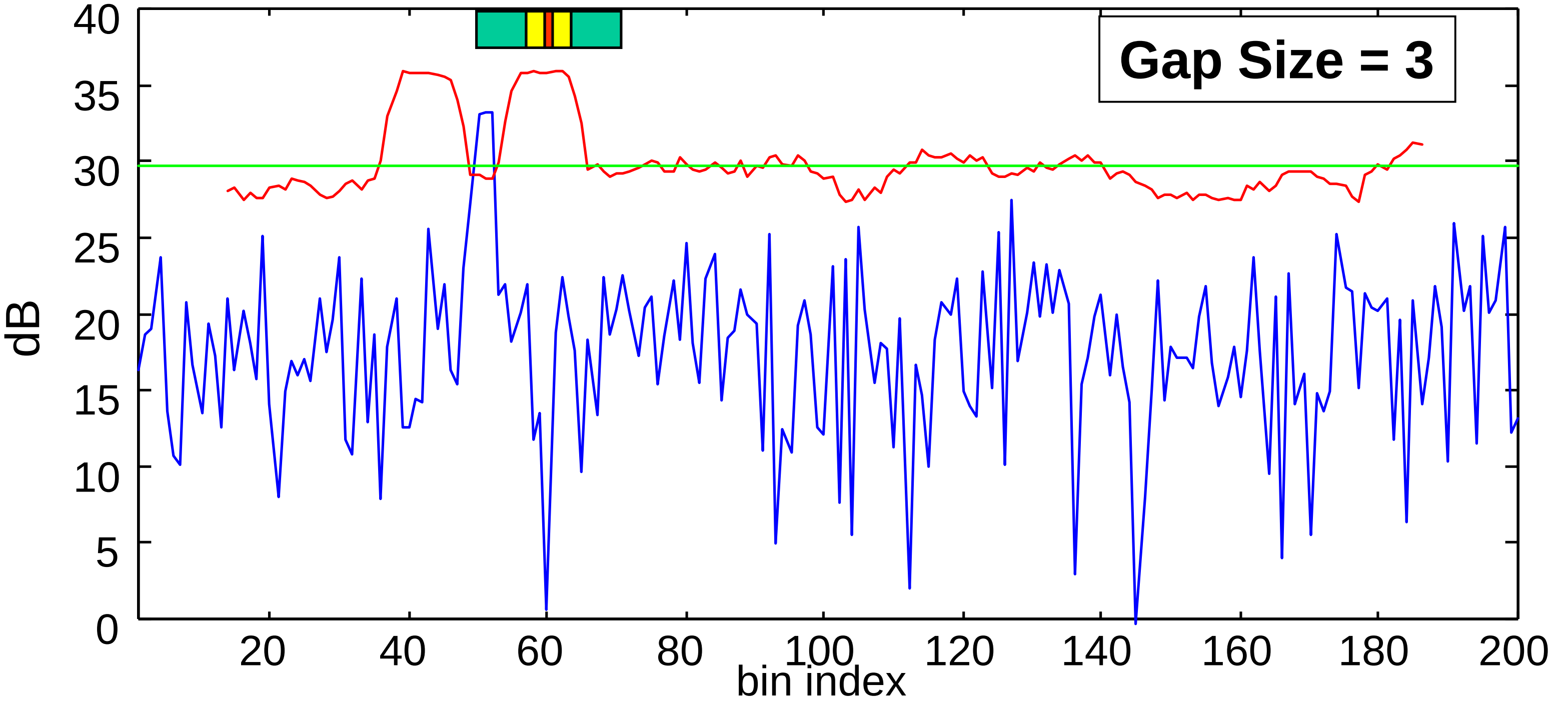
Self Masking

- If target extends over more than one data cell, it can prevent its own detection by raising the estimated “interference” level and thus the threshold
  - *lag/lead window size = 10 cells, gap size = 0 cells*



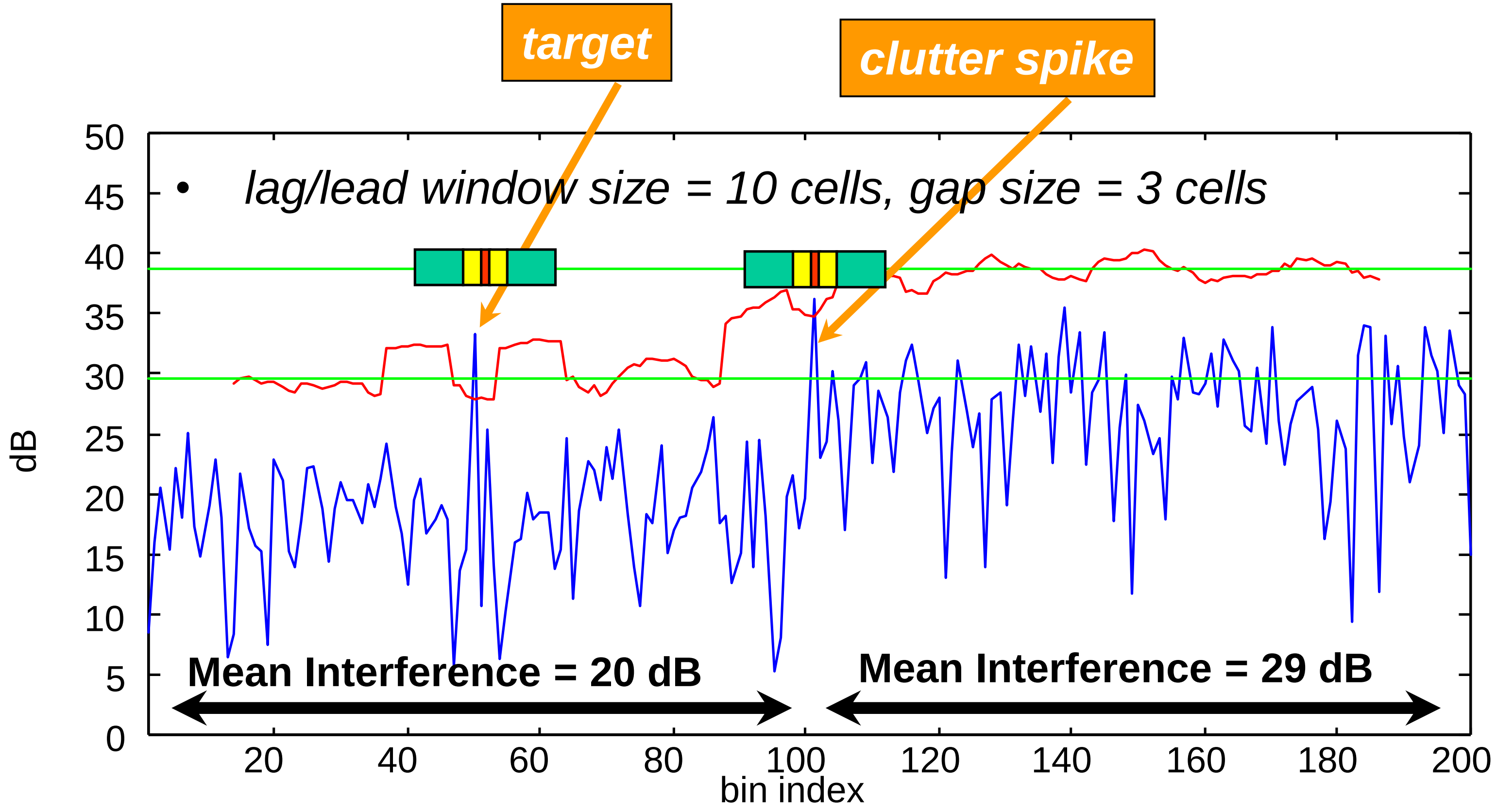
Guard Cells Combat Self Masking

- “Guard” or “gap” cells neighboring the cell under test are excluded from the mean estimate:
  - *lag/lead window size = 10 cells, gap size = 3 cells*

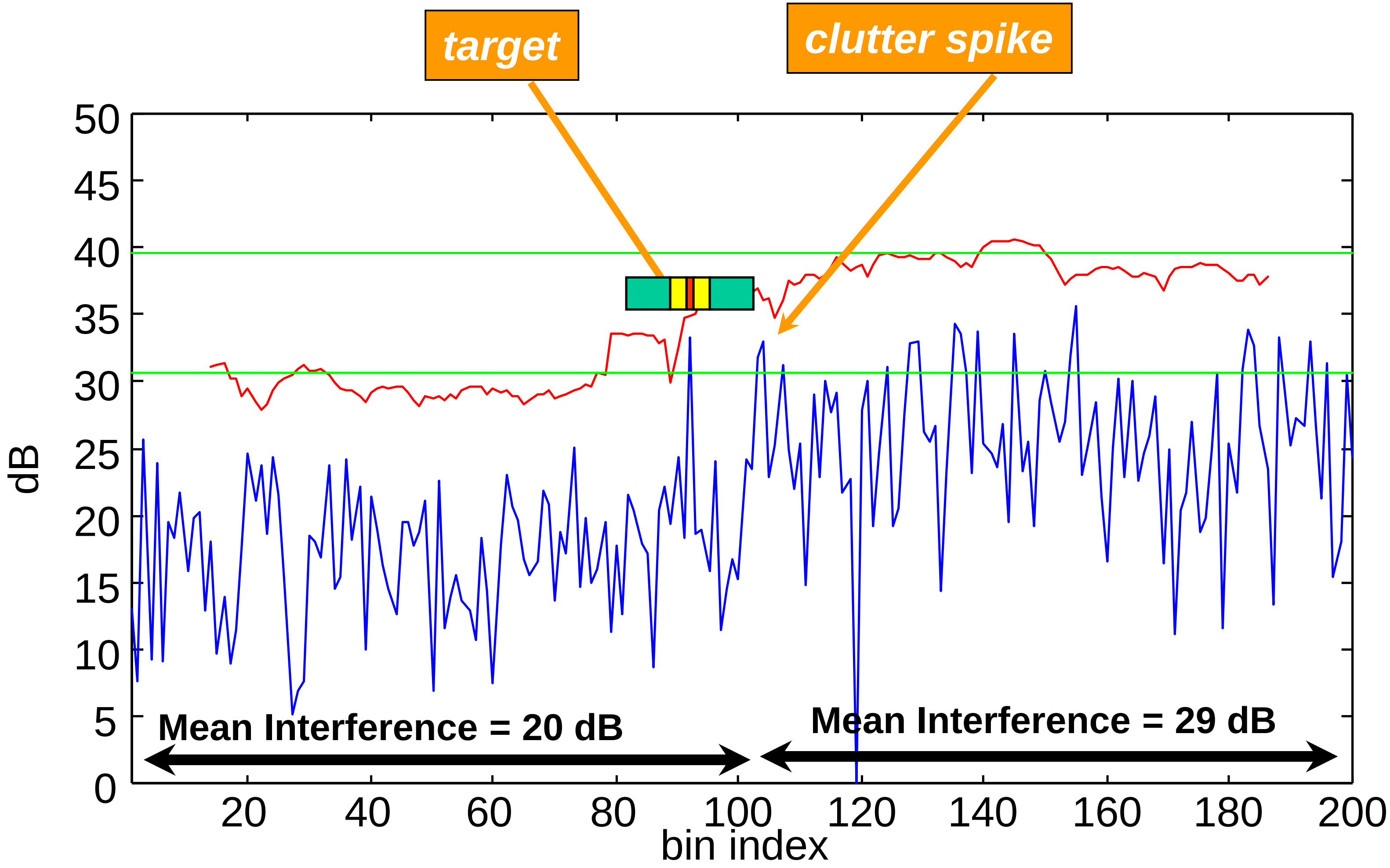


False Alarms at Clutter Edges

- Nonfluctuating target, 13 dB above the 20 dB interference
- $P_{FA} = 10^{-4}$



Clutter-Edge Masking



- *lag/lead window size = 10 cells, gap size = 3 cells*

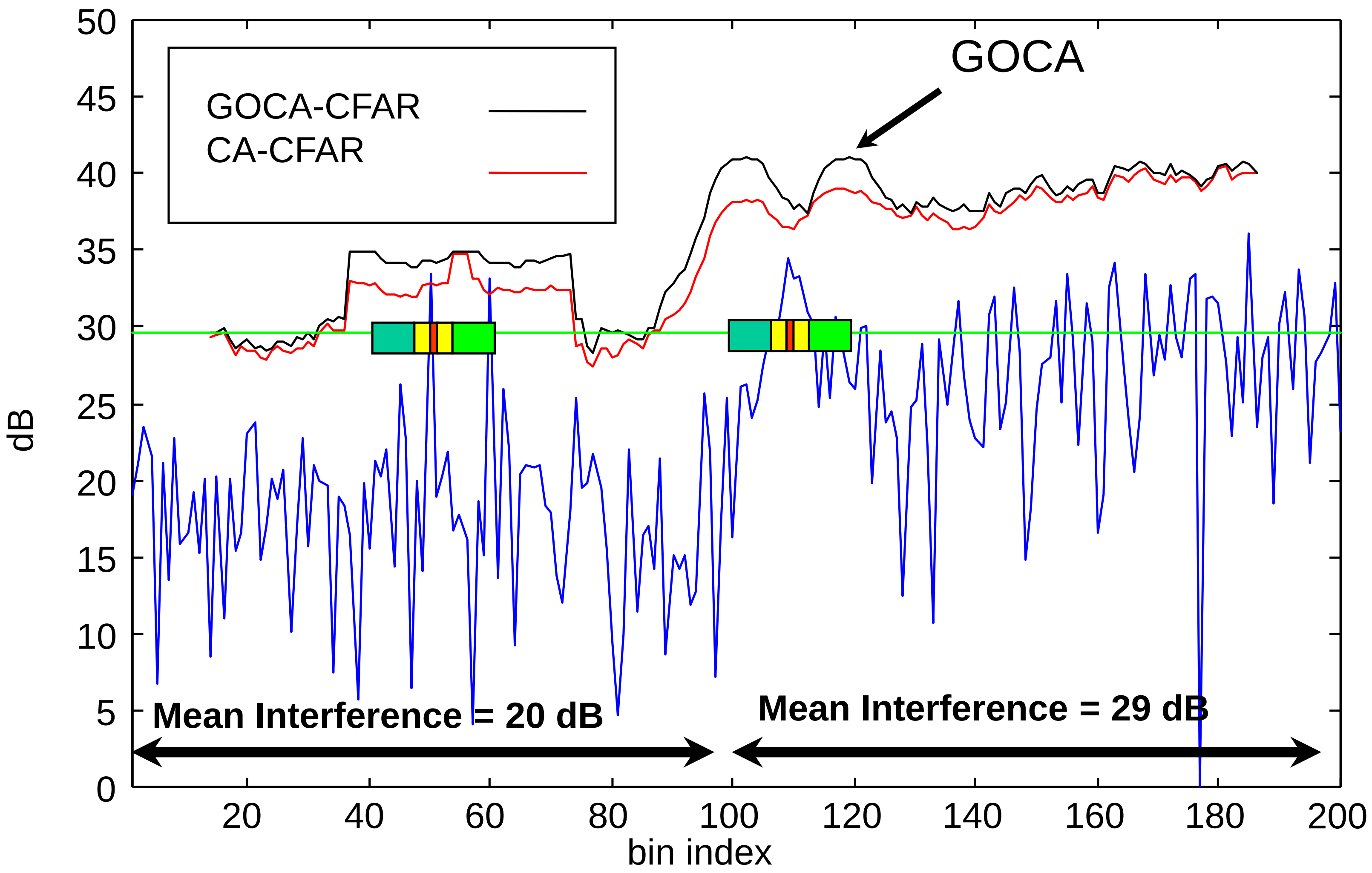
CA-CFAR Modifications for Enhanced Performance in Heterogeneous Interference

- Greater-of CA-CFAR (GOCA-CFAR)
  - Use of the greater-of (lead or lag window sum)
  - Suppresses clutter-edge false alarms
  - Degrades masking performance
  - Increases homogeneous CFAR Loss: 0.1 to 0.3 dB
- Smaller-of CA-CFAR (SOCA-CFAR)
  - Use of the smaller-of (lead or lag window sum)
  - Suppresses single-window interference masking
  - Increases clutter edge false alarms
  - Markedly increases CFAR loss for small M

From M.A. Richards, Georgia Tech

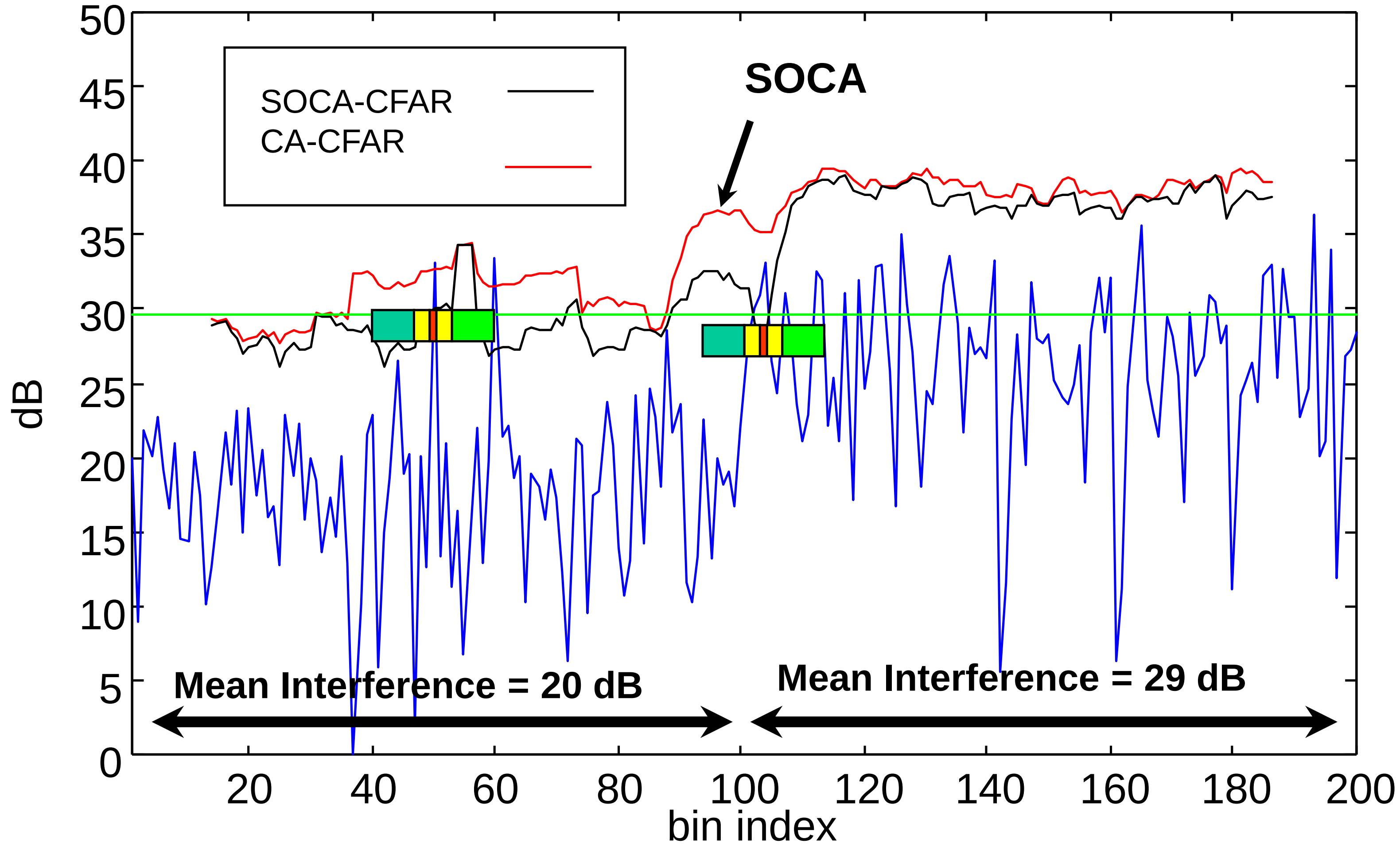
Greater-of CA-CFAR Example

- Reduced false alarms at clutter edges
- Masking is worse



Smaller-of CA-CFAR Example

- Increased false alarms at clutter edges
- Masking is reduced



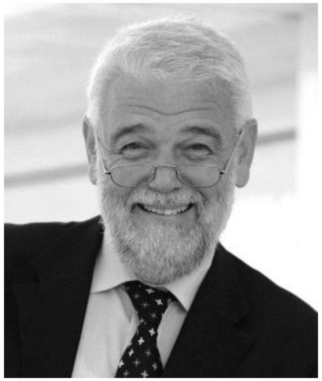
From M.A. Richards, Georgia Tech

Order Statistics CFAR (OS-CFAR)

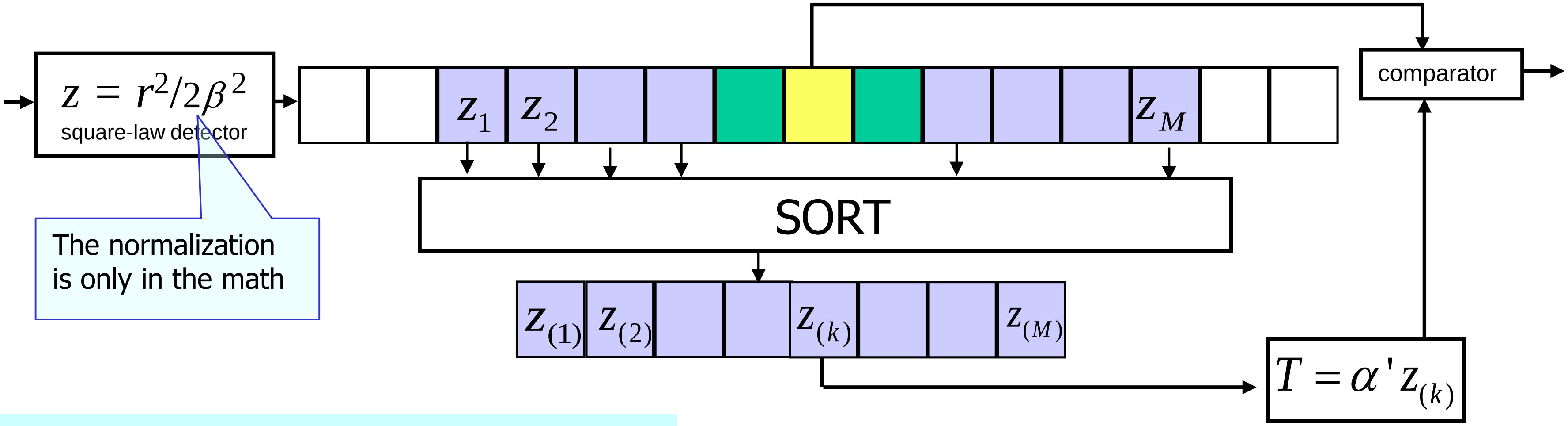
- Proposed for reducing masking degradation
- Steps:
  - Rank order the reference cells $\{z_{(1)}, \ldots, z_{(M)}\}$ by their magnitude, where $z_{(M)}$ is the largest sample
  - Estimate the interference power as equal to the k^{th} sample in the ordered sequence
    - instead of averaging all of the samples, we are choosing *one* of them to serve as the interference estimate!
  - The threshold is set as a multiple of this interference estimate, as before:

$$T = \alpha\, z_{(k)}$$

Rohling H. “Radar CFAR thresholding in clutter and multiple target situations”, *IEEE Trans. on Aerospace and Electronic Systems*, AES-19, (4), July 1983, pp. 608-621.



Order Statistics CFAR - OS CFAR



The normalization is only in the math

$$z_{(1)} \le z_{(2)} \le ......\le z_{(k)} \le .....\le z_{(M)}$$

$$p(z) = \exp\left(-z\right),\ z \ge 0,\ \text{zero elsewhere}$$

$$P_{FA}\left(T\right) = \int_{T}^{\infty} p(z)\,dz$$

$$T = \alpha\, z_{(k)}$$

$$P_{FA} = \int_{z=0}^{\infty} P_{FA}\left(T\right) p(T)\,dz$$

We need $p(T)$ which means that we need $p[z_{(k)}] = p_k(z)$

PDF of the k^{th} ranked sample

- The PDF of the k^{th} ranked sample $z_{(k)}$, is

$$p_k\left(z\right)=k\binom{M}{k}P^{k-1}\left(z\right)\left[1-P\left(z\right)\right]^{M-k}p\left(z\right)$$

$p(z)$ is the PDF of the original r.v. z, $P(z)$ is the cumulative distribution function of z

We have M samples of z ➔ $z_1, z_2, \ldots, z_i, \ldots\ldots, z_M$ They are not ordered by time or size.

We sort them ➔ $z_{(1)} \le z_{(2)} \le \ldots\ldots \le z_{(k)} \le \ldots\ldots \le z_{(M)}$

What is the PDF of the K'th ranked sample ➔ $p\left(z_{(K)}\right) \triangleq p_K\left(z\right) = ?$

The probability that any sample z_i is smaller than Z is given by the distribution function: $\Pr\left(z_i < Z\right) = P\left(Z\right)$

For the j'th ranked sample to be the *highest* sample that is smaller than Z, there must be exactly j samples (out of a total of M samples) that are smaller than Z. This is the same as the probability of j hits out of M tries, when the probability of a hit is $P(Z)$. Hence

$$\Pr\left(z_1, z_2, \ldots, z_j < Z\right) = \binom{M}{j}\left[P\left(Z\right)\right]^j\left[1 - P\left(Z\right)\right]^{M-j}$$

For the k'th ranked sample to be smaller than Z it does not have to be the *highest* sample that is smaller than Z. If any higher ranked sample is the highest sample that is smaller than Z, then this implies that the k'th ranked sample is also smaller than Z. Thus, the overall probability that the k'th ranked sample $z_{(k)}$ is smaller than Z is given by the sum of the probabilities that $z_{(k)}, z_{(k+1)}, \ldots, z_{(M)}$ are each the highest-ranked sample that is smaller than Z. Namely

$$P_K\left(Z\right) = \Pr\left(z_{(K)} < Z\right) = \sum_{j=K}^{M} \Pr\left(z_1, z_2, \ldots, z_j < Z\right)$$

$$P_K(Z)=\sum_{j=K}^{M}\binom{M}{j}\left[P(Z)\right]^{j}\left[1-P(Z)\right]^{M-j}$$

From the distribution function we get the density function by differentiation: $p_k(Z)=\dfrac{d\,P_k(Z)}{dZ}$

$$p_K(Z)=\sum_{j=K}^{M}\binom{M}{j}j\left[P(Z)\right]^{j-1}\left[1-P(Z)\right]^{M-j}p(Z)$$
$$-\sum_{j=K}^{M}\binom{M}{j}(M-j)\left[P(Z)\right]^{j}\left[1-P(Z)\right]^{M-j-1}p(Z)$$

Taking the K'th term out of the first sum, we get:

$$p_K(Z)=\binom{M}{K}K\left[P(Z)\right]^{K-1}\left[1-P(Z)\right]^{M-K}p(Z)$$
$$+\sum_{j=K+1}^{M}\frac{M!\,j}{(M-j)!\,j!}\left[P(Z)\right]^{j-1}\left[1-P(Z)\right]^{M-j}p(Z) \quad \Leftarrow \quad \frac{M!\,j}{(M-j)!\,j!}=\frac{M!}{(M-j)!\,(j-1)!}$$
$$-\sum_{j=K}^{M}\frac{M!\,(M-j)}{(M-j)!\,j!}\left[P(Z)\right]^{j}\left[1-P(Z)\right]^{M-j-1}p(Z) \quad \Leftarrow \quad \frac{M!\,(M-j)}{(M-j)!\,j!}=\frac{M!}{(M-j-1)!\,j!}$$

The last term in the lower sum, when $j=M$, is zero. Hence we can terminate the sum earlier:

$$
\begin{aligned}
p_K(Z) &= \binom{M}{K} K\left[P(Z)\right]^{K-1}\left[1-P(Z)\right]^{M-K} p(Z) \\
&+\sum_{j=K+1}^{M}\frac{M!}{(M-j)!(j-1)!}\left[P(Z)\right]^{j-1}\left[1-P(Z)\right]^{M-j} p(Z) \quad \Longleftarrow \\
&-\sum_{j=K}^{M-1}\frac{M!}{(M-j-1)!\,j!}\left[P(Z)\right]^{j}\left[1-P(Z)\right]^{M-j-1} p(Z)
\end{aligned}
$$

$$
\begin{aligned}
i &= j-1 \\
j &= i+1
\end{aligned}
$$

$$
\begin{aligned}
p_K(Z) &= \binom{M}{K} K\left[P(Z)\right]^{K-1}\left[1-P(Z)\right]^{M-K} p(Z) \\
&\left.\begin{aligned}
&+\sum_{i=K}^{M-1}\frac{M!}{(M-i-1)!\,i!}\left[P(Z)\right]^{i}\left[1-P(Z)\right]^{M-i-1} p(Z) \\
&-\sum_{j=K}^{M-1}\frac{M!}{(M-j-1)!\,j!}\left[P(Z)\right]^{j}\left[1-P(Z)\right]^{M-j-1} p(Z)
\end{aligned}\right\} = \mathbf{0}
\end{aligned}
$$

$$
\boxed{p_K(Z) = K\binom{M}{K}\left[P(Z)\right]^{K-1}\left[1-P(Z)\right]^{M-K} p(Z)}
$$

PDF of the Threshold

- For square law detector (therefore exponential pdf interference)

$$p(z) = e^{-z}, \quad P(z) = 1 - e^{-z}$$

$$p_k(z) = k\binom{M}{k}\left[e^{-z}\right]^{M-k+1}\left[1-e^{-z}\right]^{k-1}$$

- Because $T = \alpha z$, we have

$$p(T) = \frac{k}{\alpha}\binom{M}{k}\left[e^{-T/\alpha}\right]^{M-k+1}\left[1-e^{-T/\alpha}\right]^{k-1}$$

From M.A. Richards, Georgia Tech

Average P_{FA}

- Put the pieces together:

$$P_{FA}(T) = \int_T^{\infty} e^{-z} dz = e^{-T}$$

$$p(T) = \frac{k}{\alpha}\binom{M}{k}\left[e^{-T/\alpha}\right]^{M-k+1}\left[1-e^{-T/\alpha}\right]^{k-1}$$

$$P_{FA} = \int_0^{+\infty} P_{FA}(T)\, p(T)\, dT$$

$$= \int_0^{+\infty} e^{-T} \frac{k}{\alpha}\binom{M}{k}\left[e^{-T/\alpha}\right]^{M-k+1}\left[1-e^{-T/\alpha}\right] dT$$

OS-CFAR P_{FA}

- The result is, for integer α,

$$P_{FA} = k \frac{M!}{k!(M-k)!} \frac{(k-1)!\,(\alpha+M-k)!}{(\alpha+M)!}$$

(replace **!** *with gamma function for non-integer* α*)*

or

$$P_{FA} = \prod_{i=1}^{k} \left(1 + \frac{\alpha}{M+1-i}\right)^{-1}$$

P_{FA} depends only on k, M, and α therefore CFAR!

For Rayleigh background (and noise)
and for Rayleigh target

$$P_D\left(\overline{SNR},\alpha\right)=P_{FA}\left(\alpha_D\right)$$

$$\alpha_D=\left(\frac{\alpha}{1+\overline{\overline{SNR}}}\right)$$

Applies also in OS-CFAR, implying:

$$P_{FA}=\prod_{i=1}^{k}\left(1+\frac{\alpha}{M+1-i}\right)^{-1}$$

$$P_D=\prod_{i=1}^{k}\left(1+\frac{\alpha_D}{M+1-i}\right)^{-1}$$

M – the number of reference cells, k – the rank of the representative cell

Example Values of α

- α is the scale factor which multiplies the k^{th} sample to determine the threshold T
- Example for M = 16 and P_{FA} = 10^{-6}:

k	2	4	6	8	10	12	14	16
α	15,476	443	120	56.6	32.9	20.9	13.7	8.3

Characteristics of OS-CFAR

- Effective masking mitigation if number of interference-contaminated reference cells is less than M-k.
- Provides good CFAR loss trade-off
  - Homogenous interference: 1 dB more than CA-CFAR for same M
  - Heterogeneous interference: minimal masking degradation
- Provides fair clutter-edge false-alarm performance
- Rule-of-thumb: select $k = 0.75M$
- **Major penalty** – the computational complexity of sorting

Censored CA-CFAR (CCA-CFAR)

- Attempts to combine attributes of CA-CFAR and OS-CFAR
  - Masking mitigation
  - Efficient interference estimation
- Implementation
  - Rank order reference window
  - Edit N_H highest and N_L Lowest Samples
  - Compute mean of remaining samples
- Comparison to OS-CFAR
  - Performance is generally similar
  - Highly dependent upon specific scenario

J.A. Ritcey: "Performance analysis of the censored mean-level detector", *IEEE Trans. on AES*, vol. 22 (4), July 1986, pp. 443-454

Censored CA-CFAR

$$z_{(1)} \le z_{(2)} \le \ldots\ldots \le z_{(k)} \le \ldots\ldots \le z_{(M)}$$

$$z_{TH} = \alpha \frac{1}{k}\left[\sum_{i=1}^{k} z_{(i)} + (M-k) z_{(k)}\right]$$

The performances of Censored CA–CFAR (when there are no interferences in any of the reference cells) is identical to the performances of a CA-CFAR with only k reference cells (and no interferences in any of those cells). Hence α is calculated according to CA-CFAR with k reference cells.

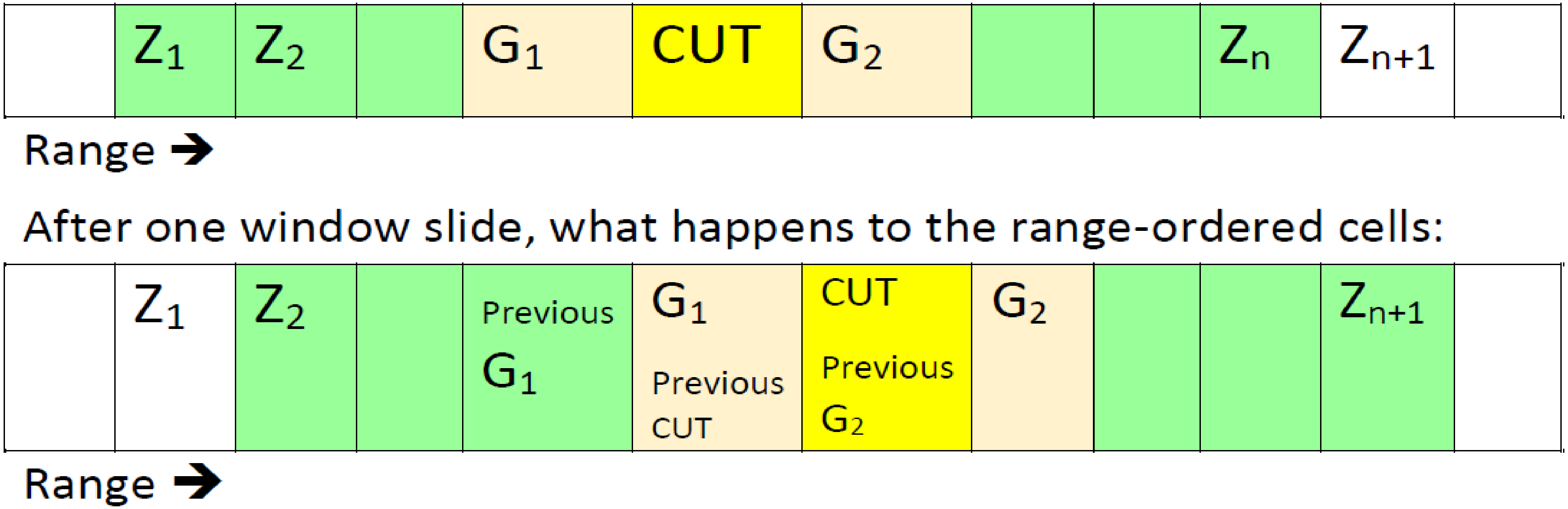
$$\alpha = k\left(P_{FA}^{-1/k} - 1\right)$$

CA-CFAR collapses if there is strong interference in any of its reference cells. The performances of Censored CA-CFAR degrade graciously as long as the number of reference cells with interferences $\le M\text{-}k$

Computational complexity of sorting

The average complexity of sorting an n element window changes between $O(n^2)$ (e.g., Bubble-sort) to $O(n \log_2 n)$ (e.g., Merge-sort or Quick-sort).

- In CFAR the sorting is repeated after each slide of the CUT.
- This can be used to update the sorting, rather than perform a new independent sort after each slide
- Sorting update is computationally simpler by a factor of $n/4$ → $O(4 \log_2 n)$



	Z_1	Z_2		G_1	CUT	G_2			Z_n	Z_{n+1}	

Range →

After one window slide, what happens to the range-ordered cells:

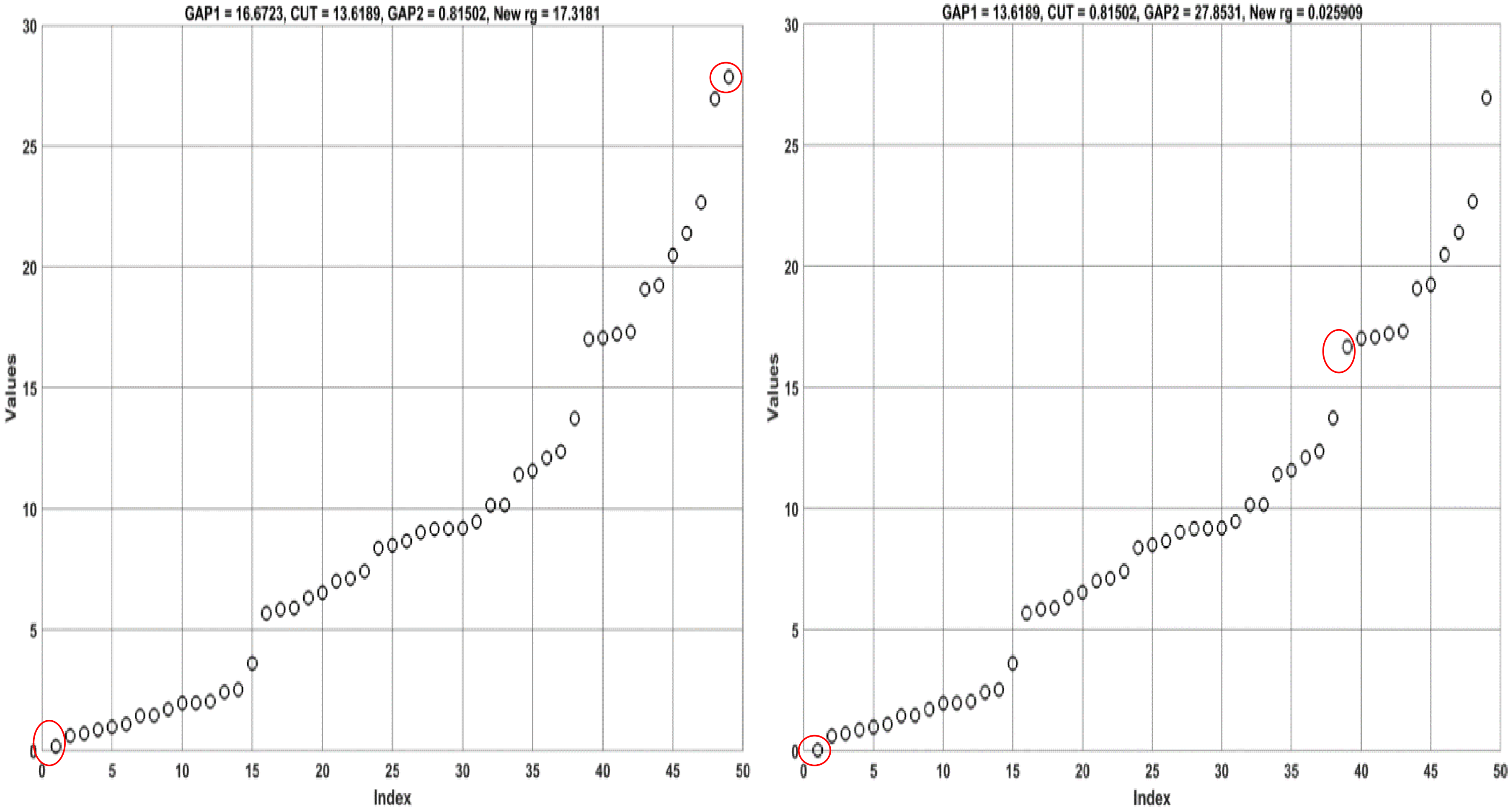
	Z_1	Z_2		Previous G_1	G_1 Previous CUT	CUT Previous G_2	G_2			Z_{n+1}	

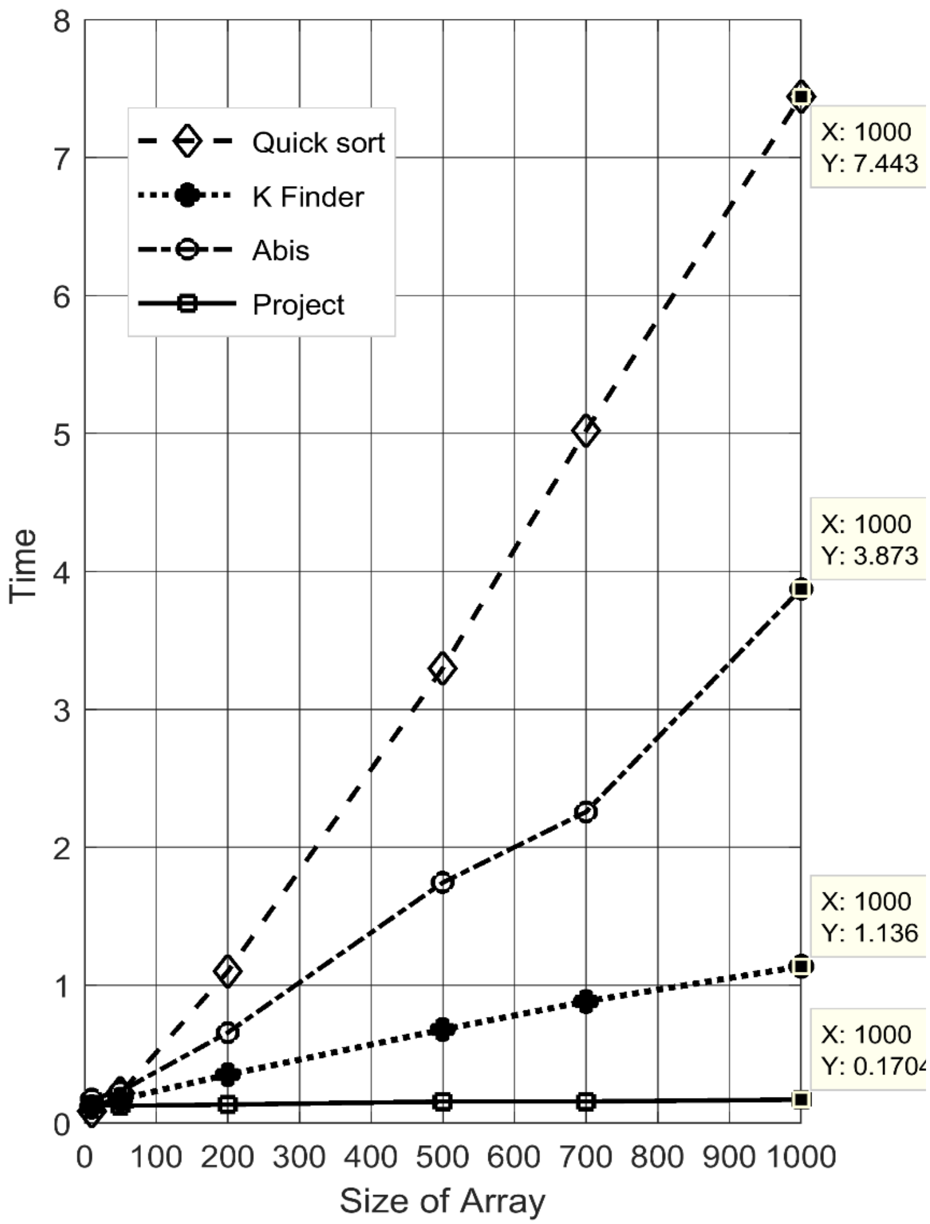
Range →

A slide of the CUT implies 4 changes in the window of reference cells

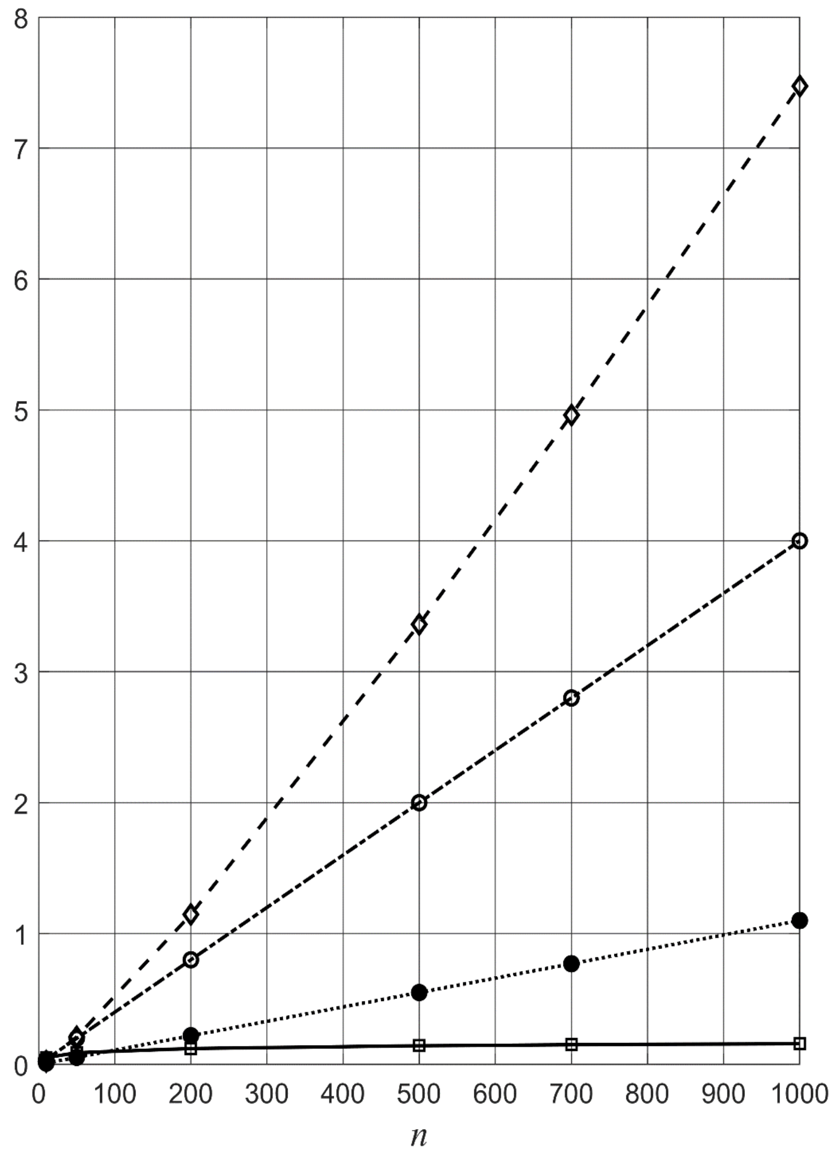
Results of two consecutive sorts ($n = 50$)

Help understand the sense of **sort update**





From Joseph Didi thesis (2018)

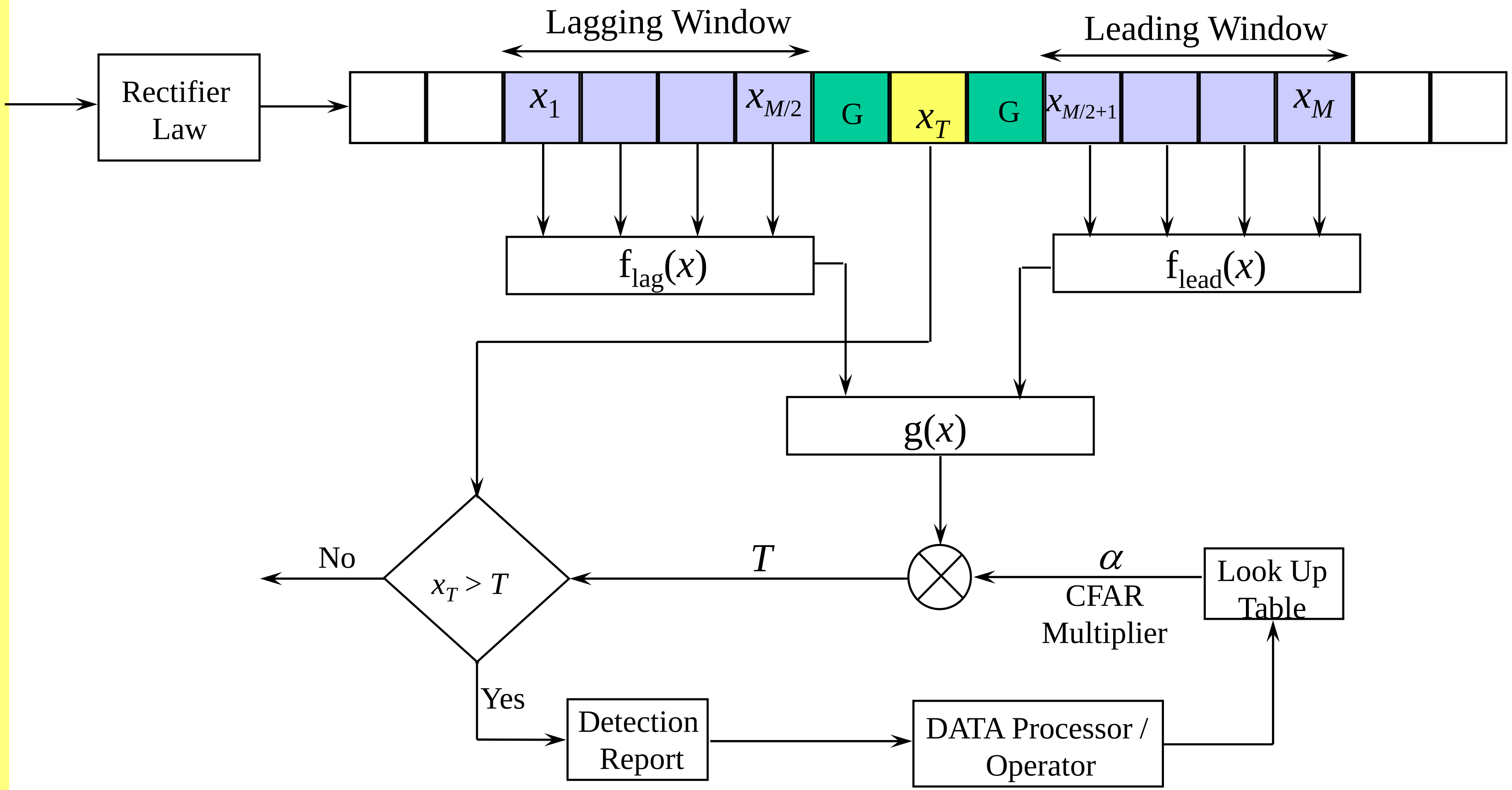


Four complexity graphs (in order of increasing slope)

$4\log_2(n)/250$, $n/909$, $n/250$, $n\log_2(n)/1330$.

From M.A. Richards, Georgia Tech

General CFAR Processor



Summary of CFAR

Algorithm	f(x)	g(x)	Comments
CA-CFAR	N/A	$\sum_{i=1}^{N} x_i$	1. Unbiased, minimum-variance estimation for homogeneous interference. 2. Subject to masking and clutter-edge false alarms.
GOCA-CFAR	$\sum_{i=1}^{N/2} x_i$	Greater of (f_{lead}(x), f_{lag}(x))	1. Suppressed clutter-edge false alarms. 2. Degraded masking performance.
SOCA-CFAR	$\sum_{i=1}^{N/2} x_i$	Smaller of (f_{lead}(x), f_{lag}(x))	1. Improved masking performance. 2. Degraded clutter-edge false alarms.
OS-CFAR	N/A	$\Re(x)\vert k$	1. Improved masking performance. 2. Some clutter-edge false alarm improvement.
CCA-CFAR	N/A	$\sum_{n=L}^{H} \Re(n) + C_0$	1. Improved masking performance. 2. Some clutter-edge false alarm improvement.
Alternate Rectification Laws			
E-CA-CFAR	Discard X_i if X_i > Xmax		1. Improved masking performance. 2. Subject to strong clutter false alarms.
L-CA-CFAR	$\mathrm{Log}_b(X)$		1. Improved masking performance. 2. Subject to clutter-edge false alarms.

$\Re(\underline{X})$: Rank - order input vector $\underline{X}$.

Threshold PDF in CCA and OS CFAR

$$p(z)=e^{-z}$$

$$P_{FA}=\prod_{i=1}^{k}\left(1+\frac{\alpha}{M+1-i}\right)^{-1}$$

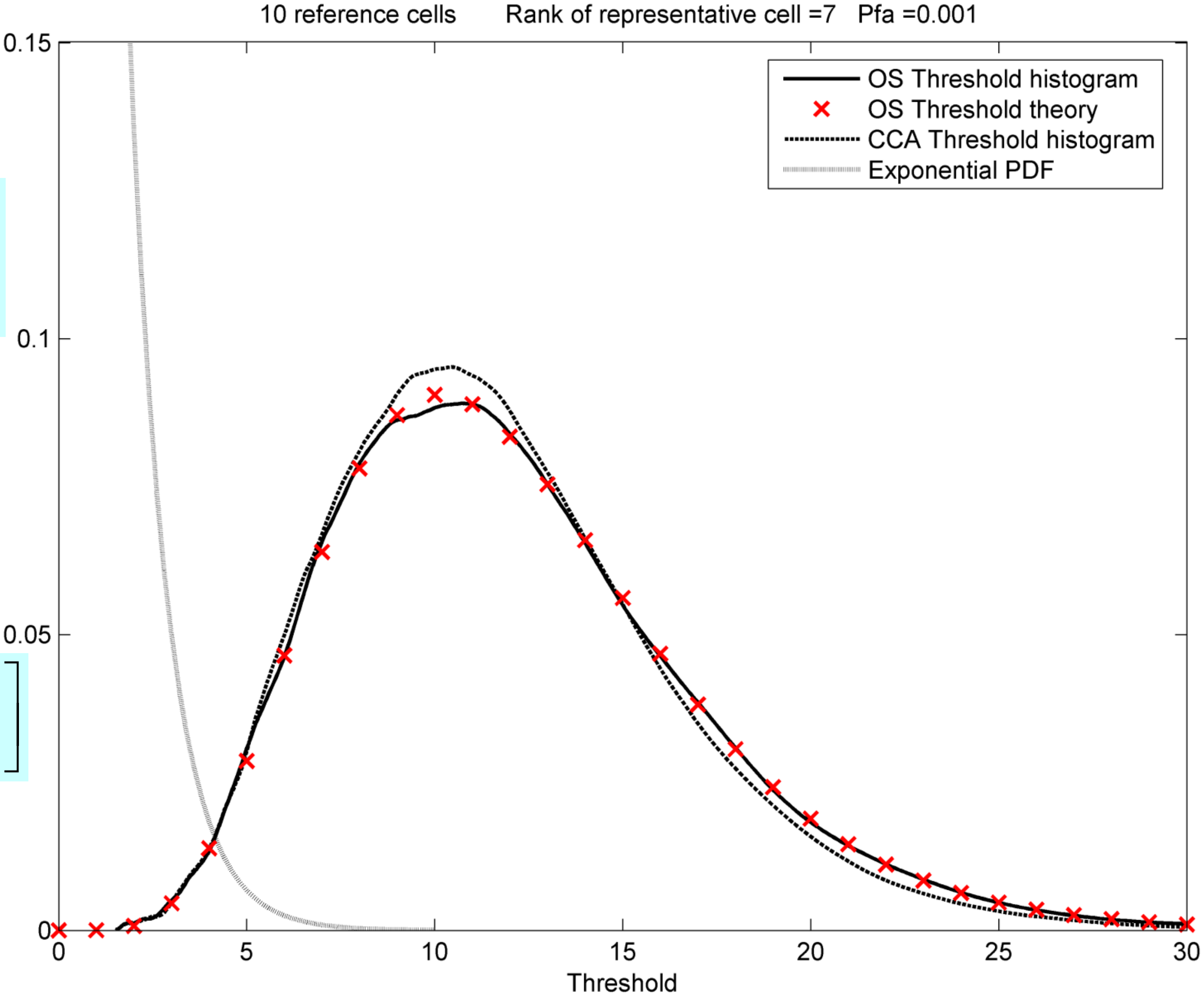
$$T=\alpha\, z_{(k)}$$

OS

$$\alpha=k\left(P_{FA}^{-1/k}-1\right)$$

CCA

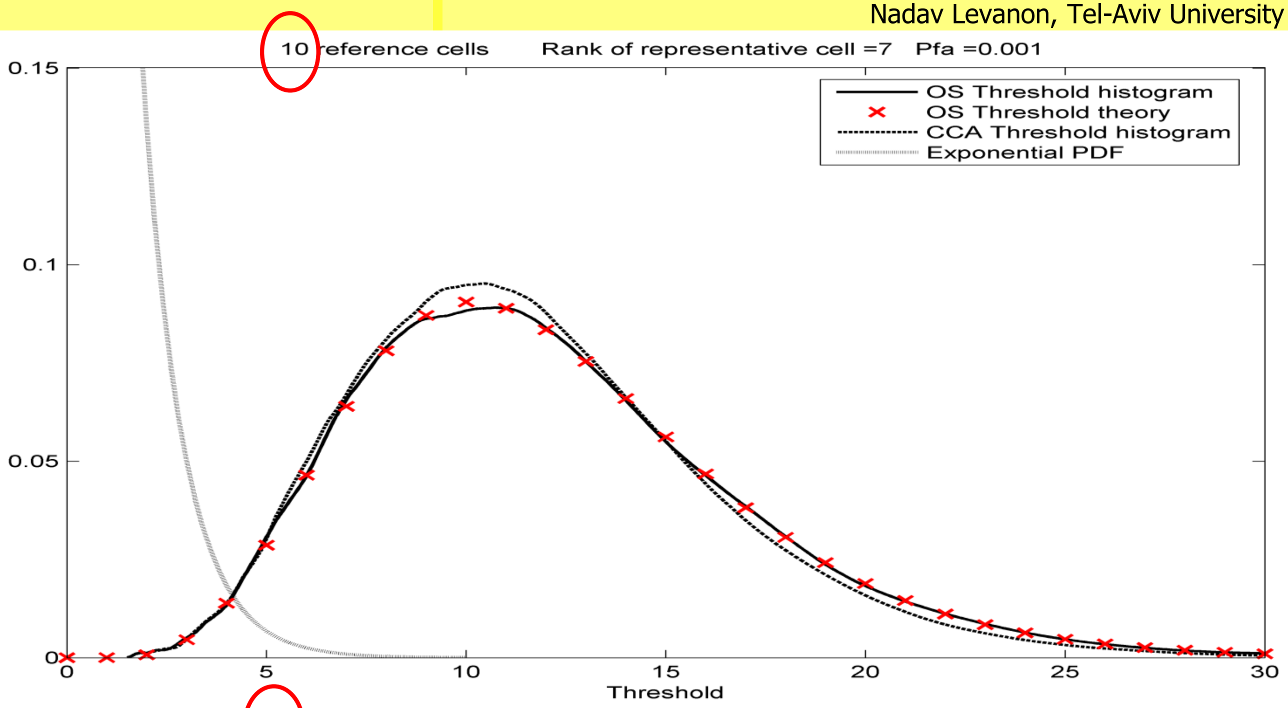
$$z_{TH}=\alpha\frac{1}{k}\left[\sum_{i=1}^{k}z_{(i)}+(M-k)z_{(k)}\right]$$



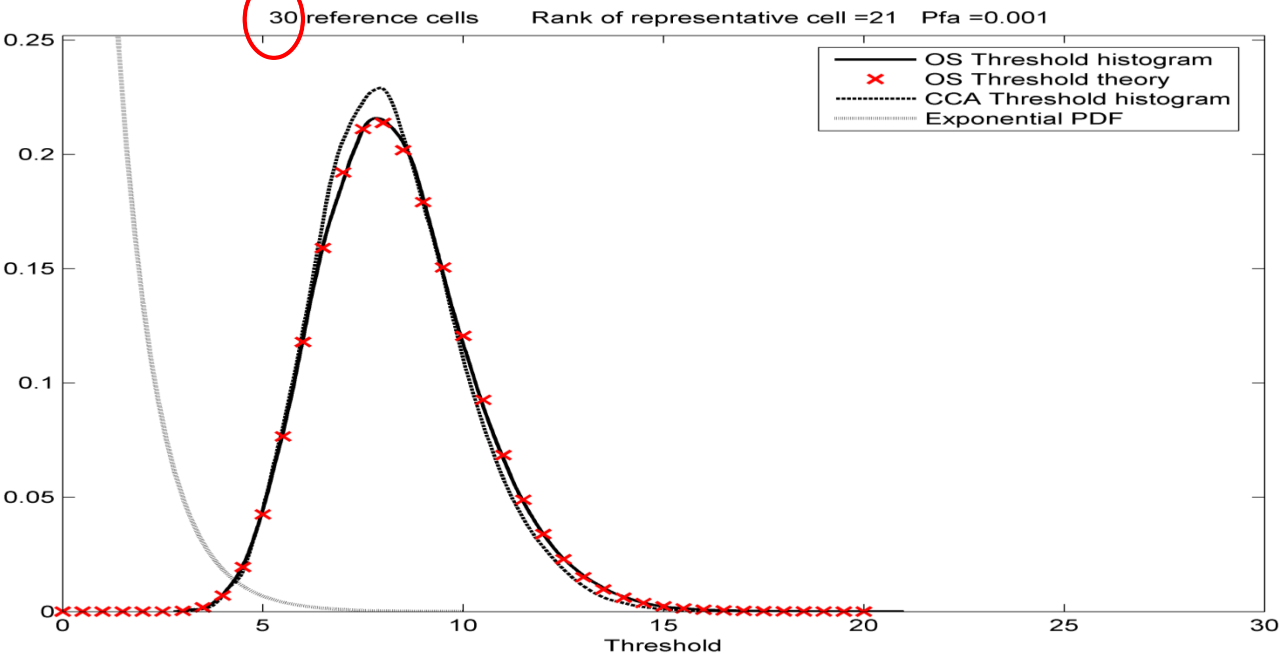
$$p(T)=\frac{k}{\alpha'}\binom{M}{k}\left[e^{-T/\alpha'}\right]^{M-k+1}\left[1-e^{-T/\alpha'}\right]^{k-1}$$

OS Theoretical

```
>> cfar_simul
 No. of reference cells
(typ=10 or 30) = ?  10
 Representative cell
(typ=7 or 21) = ?  7
 seed = ? 964
SNR [dB]  single pulse
(typ=16) = ? 16
number of trials
(typ = 40000) = ?  40000

Pfa_os  = 9.7500e-004

Pd_os   = 0.7473

Pfa_cca = 9.7500e-004

Pd_cca  = 0.7534
```

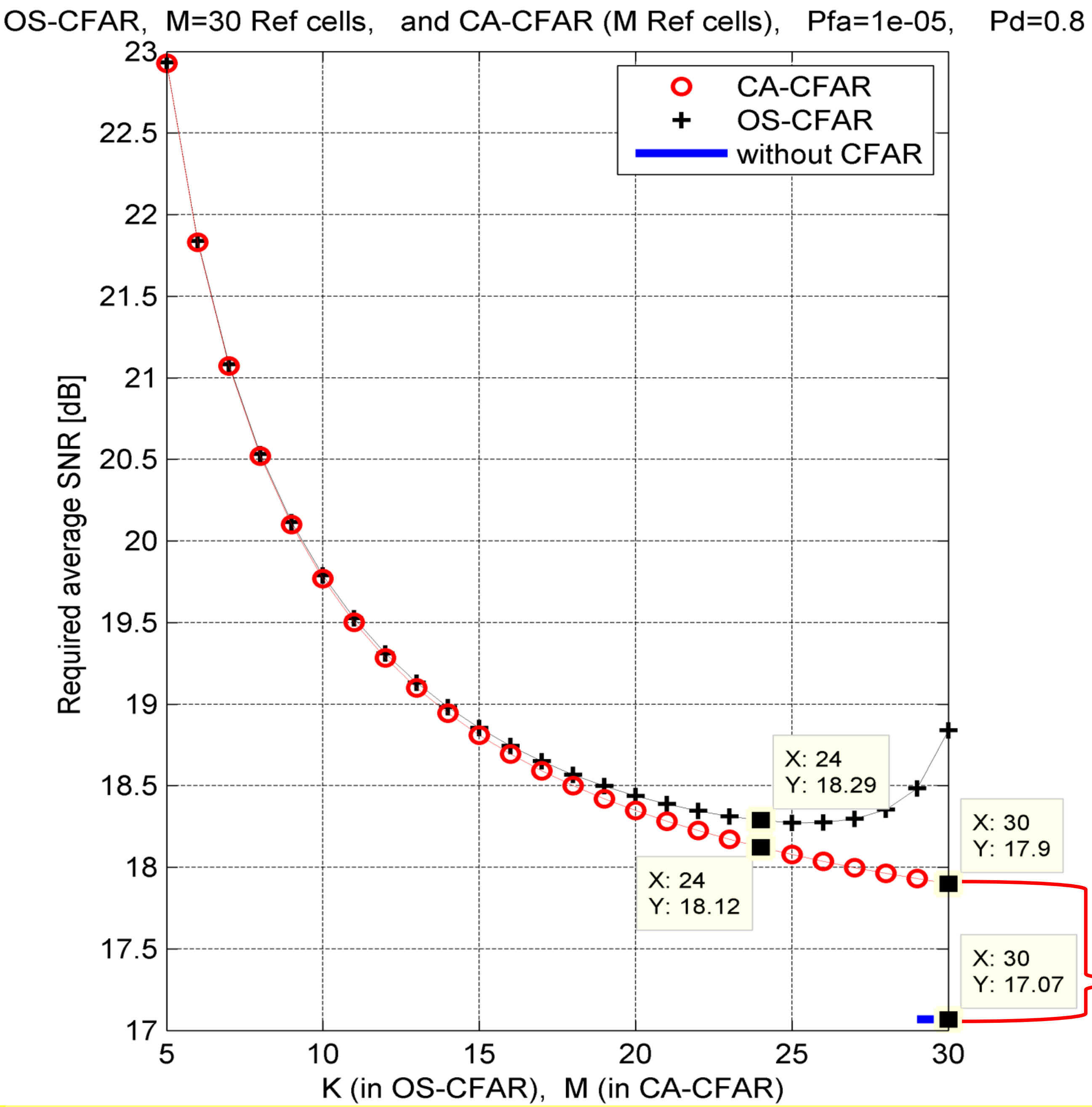


```
>> cfar_simul_30
 No. of reference cells
(typ=10 or 30) = ?  30
 Representative cell
(typ=7 or 21) = ?  21
 seed = ? 964
SNR [dB]  single pulse
(typ=16) = ? 16
number of trials
(typ = 40000) = ? 20000

Pfa_os  = 0.0010

Pd_os   = 0.8172


Pfa_cca = 1.0000e-003

Pd_cca  = 0.8196
```



CA vs. OS CFAR

performance comparison



$$\text{CFAR LOSS}_{[dB]} \underset{M\ge 16}{\approx} -\frac{5}{M}\log_{10}\left(P_{FA}\right)$$

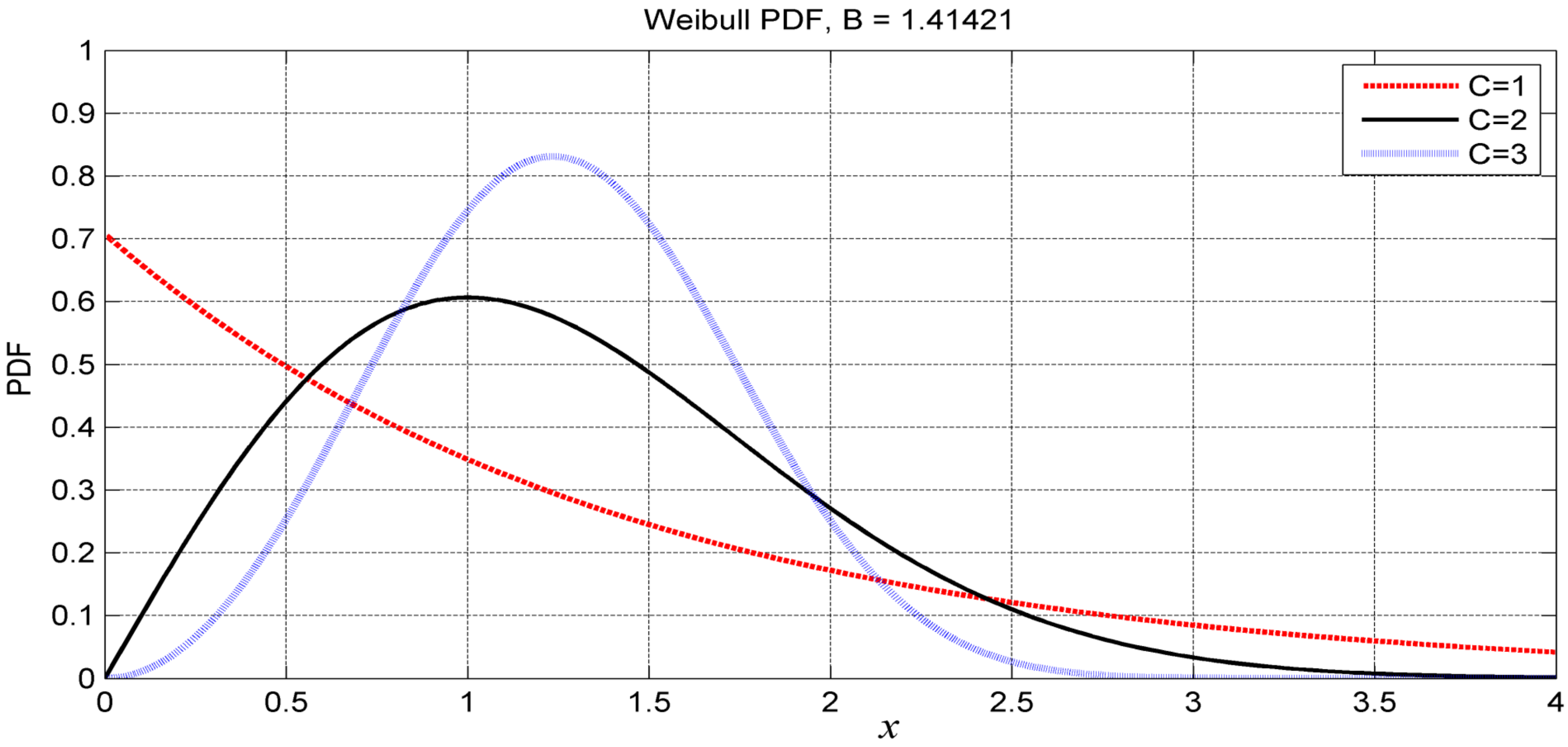
$$\text{CFAR LOSS}\Big|_{M=30} \approx 0.83\text{dB}$$

What happens if the background statistics is not Rayleigh, as expected?

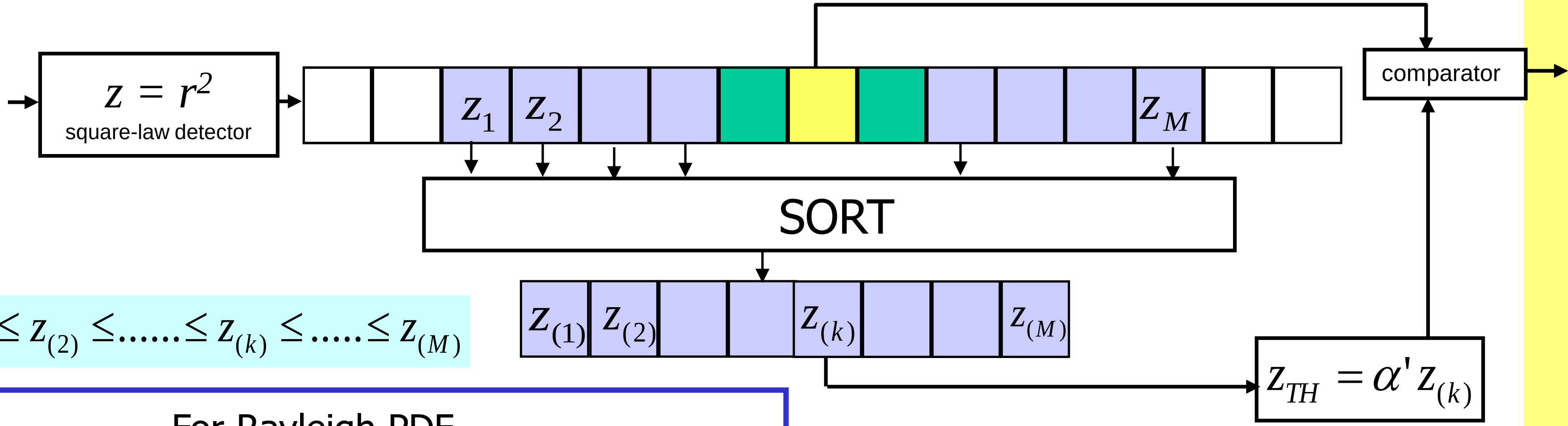
Will be demonstrated using Weibull PDF

$$p_x(x) = \frac{C}{B}\left(\frac{x}{B}\right)^{C-1} \exp\left[-\left(\frac{x}{B}\right)^{C}\right], \quad 0 \le x, \quad \text{zero elswhere}$$

C=1 ➔ Exponential PDF
C=2 ➔ Rayleigh PDF



Order Statistics CFAR - OS CFAR



$$z_{(1)} \le z_{(2)} \le \ldots\ldots \le z_{(k)} \le \ldots\ldots \le z_{(M)}$$

For Rayleigh PDF

$$p_r(r) = \frac{r}{\beta^2} \exp\left(\frac{-r^2}{2\beta^2}\right), \quad r > 0$$

$$T = \alpha\, z_{(k)}$$

$$P_{FA} = \prod_{i=1}^{k}\left(1 + \frac{\alpha}{M+1-i}\right)^{-1}$$

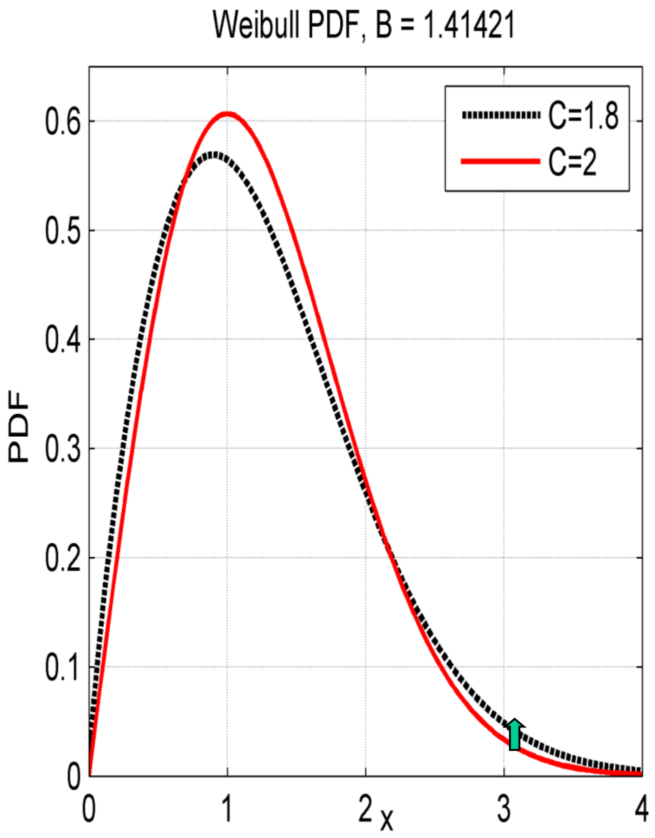
For Weibull PDF

$$p_r(r) = \frac{C}{B}\left(\frac{r}{B}\right)^{C-1} \exp\left[-\left(\frac{r}{B}\right)^{C}\right], \; r \ge 0$$

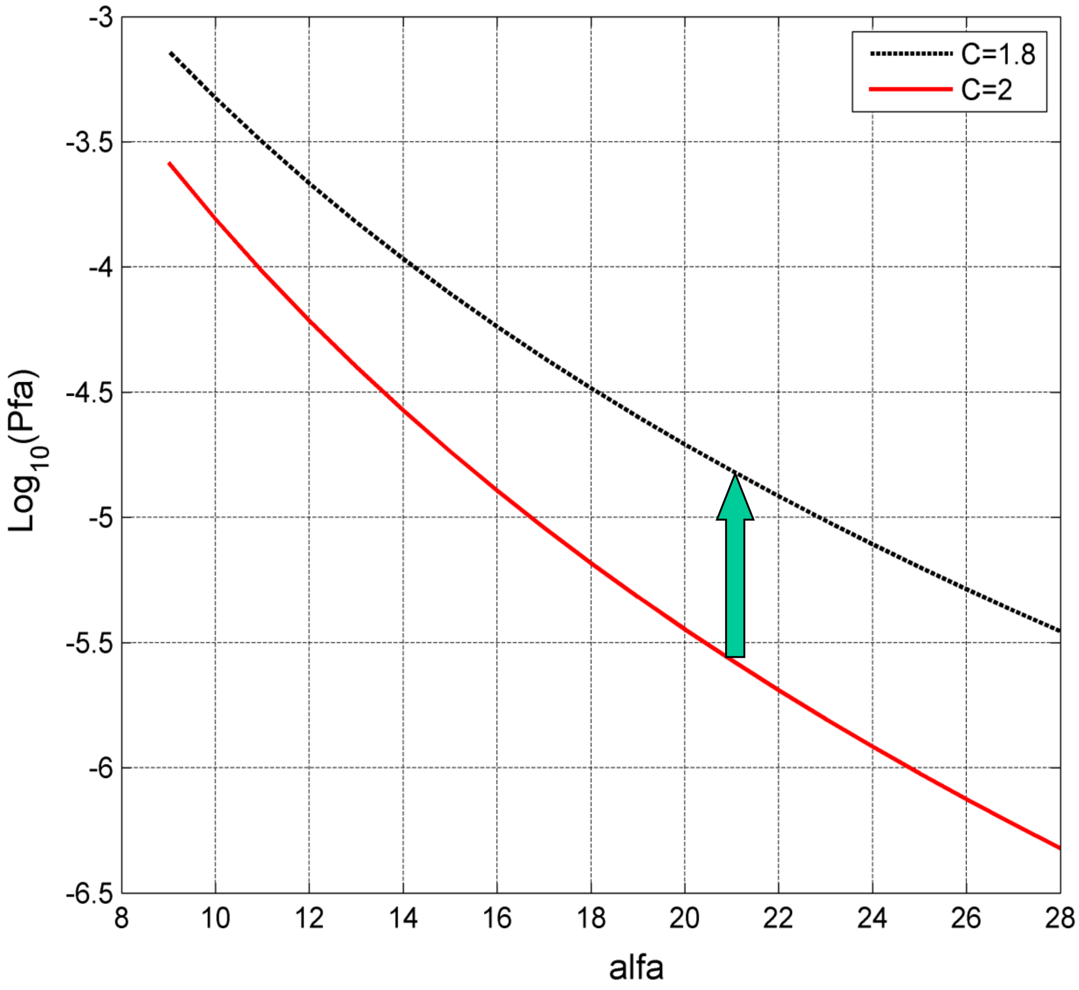
$$T = \alpha_c\, z_{(k)}$$

$$P_{FA} = \prod_{i=1}^{k}\left(1 + \frac{\alpha_c^{C/2}}{M+1-i}\right)^{-1}$$

$$C = 2 \Rightarrow \alpha = \alpha_c$$



P_{fa} increased because C was smaller (i.e., PDF tail higher) than assumed.



$M = 10,\ k = 7$

It is possible to estimate Weibull's 2-parameters: B and C.

For example, by using 2 representative sorted cells.

However the estimate is poor and the CFAR loss is likely to be higher than if we estimate only B and use the lowest expected C to calculate α .

The classical CFAR for a 2-parameter background (specifically for log-normal or Weibull) is the **log-*t*** CFAR suggested by G.B. Goldstin[*] in 1973.

---

[*] Goldstein, G. B. "False-alarm regulation in Log-Normal and Weibull clutter," *IEEE Trans. AES*, Vol. AES-9, (1) , January 1973, pp. 84-92