Periodic Ambiguity Function of CW Signals with Perfect Periodic Autocorrelation

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A periodic ambiguity function (PAF) is discussed which describes the response of a correlation receiver to a CW signal modulated by a periodic waveform, when the reference signal in the receiver is constructed from an integral number N, of periods T, of the transmitted signal. The PAF is a generalization of the periodic autocorrelation function, to the case of non-zero Doppler shift.

We show that the PAF of N periods is obtained by multiplying the PAF of a single period by the universal function $\sin(N\pi\nu T)/N\sin(\pi\nu T)$, where ν is the Doppler shift.

The PAF is then used to study the performance of the correlation receiver, in the presence of Doppler shift, to phase modulated signals which exhibit perfect periodic autocorrelation when there is no Doppler shift. The PAF of these signals exhibit universal cuts along the delay and Doppler axes. These cuts are functions only of T, N and the number M, of modulation bits in one period.

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I. INTRODUCTION

The periodic ambiguity function (PAF) describes the response of a correlation receiver to a continuous signal modulated by a periodic waveform with period T, when the reference signal is constructed from an integral number N of periods of the transmitted signal. Thus, the reference signal is of duration NT, (Fig. 1). The response is a function of both delay and Doppler shift. The PAF is a two-dimensional generalization of the periodic autocorrelation function, by including the effect of Doppler shift. A major difference between the periodic autocorrelation and the periodic ambiguity function, is the significance of the number of periods N. In the autocorrelation case, the response of N periods differs from the response of a single period, only by the factor N. When Doppler is present, and has to be resolved, the ability to resolve it is a function of the duration of the reference signal, and the effect of N on the response is more significant and complicated. However, the effect of N is independent of the modulation waveform and of the delay.

CW signals with periodic modulation are important because only they can yield a perfect autocorrelation. By perfect periodic autocorrelation we mean an autocorrelation of value 1 when $\tau = 0 \pmod T$ and zero elsewhere. A finite length signal inherently cannot achieve such an ideal autocorrelation. When the first (or last) bit of the signal enters (or leaves) the correlator, there is no other bit which can cancel the product to yield a zero output.

The periodic autocorrelation is the cut of the PAF at zero Doppler shift. With increasing Doppler shift this perfect behavior is gradually lost. Cuts of the PAF in parallel to the delay axis, yield the response of the correlation receiver at a given Doppler shift. Thus, the PAF is the proper function to describe the deterioration from perfect autocorrelation, as the Doppler shift is increased.

CW signals modulated by a periodic function, are used extensively in radar [1]. The fact that the transmitted signal is continuous does not imply that the portion processed in order to make a measurement or reach a decision, is infinitely long. There are physical constraints such as the illumination time, and there are processor constraints. Our discussion involves a coherent processor of finite duration NT. In order for such a processor to perform as predicted by theory, the target return signal has to be available for a duration that is a few periods longer than NT, but not infinitely long.

The same phenomena is observed in pulse radar. The transmitted signal is an infinitely long sequence of pulses. The coherent processor involves N pulses, and the target illumination is required to be longer than NT. As a matter of fact, we see that the CW signals discussed here yield performances, in the

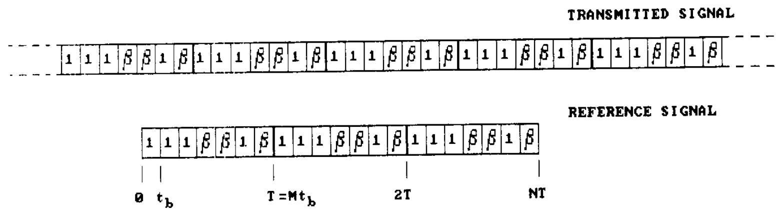


Fig. 1. Typical transmitted signal and its reference signal.

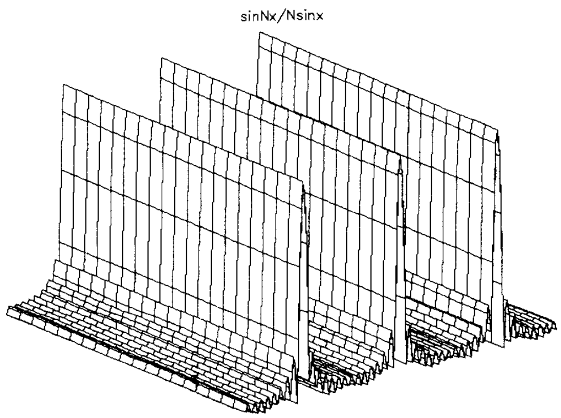


Fig. 2. The function $|\sin Nx / N \sin x|$.

delay-Doppler domain, that strongly resemble the performances of a coherent pulse train.

II. PERIODIC AMBIGUITY FUNCTION

Let the transmitted signal be a CW signal with a periodic complex envelope $u(t)$ with period T,

$$u(t) = u(t + nT), \qquad n = 0, \pm 1, \pm 2, \ldots . \tag{1}$$

Following the conventional definition of the ambiguity function [2, 3] for signals with finite energy, the single-period ambiguity function of a signal with periodic complex envelope will be [4]

$$\chi_T(\tau,\nu) \triangleq \frac{1}{T}\int_0^T u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) \times \exp(j2\pi\nu t)\, dt. \tag{2}$$

While for finite energy signals there are other forms of the ambiguity function, for periodic signals this particular form is essential to maintain the symmetry property

$$|\chi_T(-\tau,-\nu)| = |\chi_T(\tau,\nu)|. \tag{3}$$

When the reference signal is of duration NT, the response of the correlation receiver is the PAF for N periods, which, after normalization, is defined by

$$\chi_{NT}(\tau,\nu) \triangleq \frac{1}{NT}\int_0^{NT} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) \times \exp(j2\pi\nu t)\, dt. \tag{4}$$

Splitting the integral into N sections we get

$$\chi_{NT}(\tau,\nu) = \frac{1}{NT}\sum_{n=1}^{N}\int_{(n-1)T}^{nT} u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) \times \exp(j2\pi\nu t)\, dt. \tag{5}$$

Using the transformation $t = t' + (n-1)T$ yields

$$\chi_{NT}(\tau,\nu) = \frac{1}{NT}\sum_{n=1}^{N}\int_{0}^{T} u\left[t' + (n-1)T + \frac{\tau}{2}\right] \times u^*\left[t' + (n-1)T - \frac{\tau}{2}\right] \times \exp\{j2\pi\nu[t' + (n-1)T]\}\, dt'. \tag{6}$$

Because of the periodicity, expressed in (1), we get

$$\begin{aligned}
\chi_{NT}(\tau,\nu) &= \frac{1}{NT}\sum_{n=1}^{N}\exp[j2\pi\nu(n-1)T] \times \int_0^T u\left(t+\frac{\tau}{2}\right) u^*\left(t-\frac{\tau}{2}\right) \times \exp(j2\pi\nu t)\, dt \\
&= \frac{1}{N}\chi_T(\tau,\nu)\sum_{n=1}^{N}\exp[j2\pi\nu(n-1)T] \\
&= \frac{1}{N}\chi_T(\tau,\nu)\frac{1-\exp(j2\pi\nu NT)}{1-\exp(j2\pi\nu T)} \\
&= \chi_T(\tau,\nu)\frac{\sin(\pi\nu NT)}{N\sin(\pi\nu T)}\exp[j\pi\nu(N-1)T].
\end{aligned} \tag{7}$$

Hence

$$|\chi_{NT}(\tau,\nu)| = |\chi_T(\tau,\nu)|\left|\frac{\sin(\pi\nu NT)}{N\sin(\pi\nu T)}\right|. \tag{8}$$

Equation (8) is an important result, indicating the effect of using a reference signal consisting of N modulation periods. Note that the effect is to multiply the ambiguity function of a single period by a universal function of N and T, which is independent of the complex envelope of the signal, and which does not change with τ. An example of such a universal function is plotted in Fig. 2.

Now that we have the tool to analyse periodic signals, we apply it to a special family of CW signals, which yield perfect periodic autocorrelation, and find if and how this perfect response deteriorates due to Doppler shift.

III. TWO-VALUED PHASE CODING WITH PERFECT PERIODIC AUTOCORRELATION

Perfect periodic autocorrelation can be found in signals with periodic modulation where each period is constructed from a sequence of M bits of duration t_b. The complex envelope during one period is given by

$$u(t) = \sum_{m=1}^{M} u_m[t-(m-1)t_b], \qquad 0 \le t \le Mt_b \tag{9}$$

where

$$u_m(t) = \begin{cases} \exp(j\phi_m), & 0 \le t < t_b \\ 0, & \text{elsewhere} \end{cases}. \tag{10}$$

Note that the periodicity of the CW signal is maintained. Namely, (1) still applies, with the relation

$$T = Mt_b. \tag{11}$$

The periodic autocorrelation values of such a signal, at delays which are multiples of t_b, are given by

$$C(r) = C(rt_b) = \frac{1}{M}\sum_{m=1}^{M} u_m u_{m+r}^*, \qquad r = 0,1,2,\ldots \tag{12}$$

where

$$u_m = u_m(0). \tag{13}$$

The perfect periodic autocorrelation, which we strive to obtain, is described by

$$C(r) = \begin{cases} 1, & r = 0 \ (\mathrm{mod}\, M) \\ 0, & r \ne 0 \ (\mathrm{mod}\, M), \text{ i.e.,} \\ & r = 1,2,\ldots,M-1. \end{cases} \tag{14}$$

When there are no restrictions on the number of values that ϕ_m can take, the codes are called polyphase. There are many polyphase sequences that yield perfect periodic autocorrelation [5, 6]. Any sequence whose discrete Fourier transform yields elements of unit magnitude, is a proper sequence [7]. In other words, in order for the phase sequence $\{\phi_m\}$ to yield perfect periodic autocorrelation it must obey the Fourier transform equation

$$\frac{1}{\sqrt{M}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{M-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(M-1)} \\ \vdots & & & & \vdots \\ 1 & w^{M-1} & w^{2(M-1)} & \cdots & w^{(M-1)^2} \end{bmatrix} \begin{bmatrix} 1 \\ \exp(j\phi_1) \\ \exp(j\phi_2) \\ \vdots \\ \exp(j\phi_{M-1}) \end{bmatrix} = \begin{bmatrix} 1 \\ \exp(j\alpha_1) \\ \exp(j\alpha_2) \\ \vdots \\ \exp(j\alpha_{M-1}) \end{bmatrix} \tag{15}$$

where

$$w = \exp\left(-j\frac{2\pi}{M}\right) \tag{16}$$

and there are no restrictions on the resulting $\{\alpha_m\}$.

From an implementation point of view there is an important practical value to signals in which ϕ_m can take one of only two values. Without loss of generality, the two values can be zero and ϕ. Hence, u_m can be either 1 or β, where

$$\beta = \exp(j\phi). \tag{17}$$

We can conclude that the complex envelop of such a signal can be described by a periodic two-valued sequence $U = \{u_m\}$ having period M, where the two values are the complex numbers 1 and β.

How to construct the sequence U, and what should be the value of the phase ϕ that will yield a perfect periodic autocorrelation, is studied in the accompanied paper (Golomb [8], this issue). Golomb shows that the sequence U must correspond to a (M,k,λ) difference set D, where $u_m = 1$ for $m \in D$, and $u_m = \beta$ for $m \notin D$. The sequence U corresponds to a (M,k,λ) difference set [9], if M is the length of the sequence, k is the number of 1s and λ is a constant number of times that 1s in the original sequence coincide with 1s in the shifted sequence, for any cyclic shift. Note that all the "Maximal length linear shift register sequences" correspond to such difference sets [9].

For example, in the sequence $U = \{1\ 1\ 1\ \beta\ \beta\ 1\ \beta\}$ the 1s correspond to the set $D : \{0,1,2,5\} \bmod 7$, which is a (7,4,2) difference set.

For such a sequence, Golomb shows that the autocorrelation function is given by

$$C(0) = 1 \tag{18}$$

$$C(r \ne 0) = \frac{M - 2k + 2\lambda + 2(k-\lambda)\cos\phi}{M}. \tag{19}$$

In order to meet the ideal requirement in (14) we equate (19) to zero and obtain an expression for ϕ

$$\phi = \arccos\left[-\frac{M-2k+2\lambda}{2(k-\lambda)}\right]. \tag{20}$$

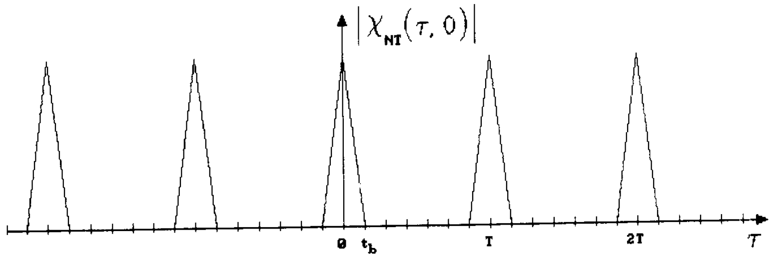


Fig. 3. Cut of ambiguity function along delay axis.

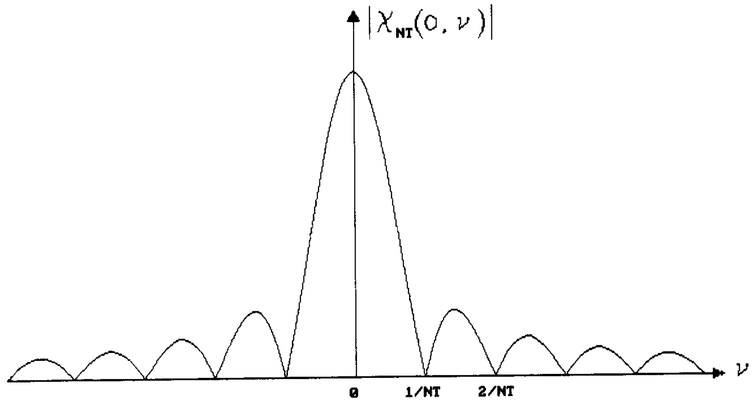


Fig. 4. Cut of ambiguity function along Doppler axis.

For the sequence of length 7 in the example above, ϕ is

$$\phi = \arccos\left(-\tfrac{3}{4}\right) = 138.59°.$$

The reader may have noticed already that this example corresponds to the Barker sequence of length 7, where the -1 elements were replaced by β. However, not all the Barker sequences lend themselves to this modification. The sequences with length 5 and 13 produce an absolute value of the argument of the arccos that is greater than 1. That means that there is no ϕ that will provide a perfect periodic autocorrelation for either of these sequences. On the other hand, there are sequences other than Barker, that do meet the requirement. In particular, all the Hadamard sequences [9] can yield perfect periodic autocorrelation. Hadamard sequences correspond to Hadamard difference sets, in which

$$M = 4n-1, \qquad k = 2n-1, \qquad \lambda = n-1, \qquad \text{where} \quad n = 1,2,\ldots. \tag{21}$$

For these sequences (when they exist)

$$\phi = \arccos\left(-\frac{2n-1}{2n}\right) = \arccos\left(-\frac{M-1}{M+1}\right). \tag{22}$$

An example is the Hadamard sequence of length 15

$$\{1\ 1\ 1\ 1\ \beta\ \beta\ \beta\ 1\ \beta\ \beta\ 1\ 1\ \beta\ 1\ \beta\}$$

which corresponds to the (15,8,4) difference set, yielding perfect periodic autocorrelation when $\phi = \arccos(-7/8) = 151.045°$. Note that as the length of the sequence increases, ϕ approaches 180°. With the help of (22) we see that the Hadamard sequence of length $M = 131$, will have perfect periodic autocorrelation when $\phi = 170°$.

IV. CUTS OF AMBIGUITY FUNCTION ALONG DELAY AND DOPPLER AXES

The information we already have permits us to determine the cuts of the periodic ambiguity function along the two major axes, without requiring further information on the signal, beyond what we already know.

The cut along the delay axis (zero Doppler) will be the perfect periodic autocorrelation as given by (14). At delays which are not multiples of t_b the autocorrelation is given by straight line segments connecting the values at multiples of t_b. (See e.g., [3, insert 5A]). A typical cut of the PAF along the delay axis is given in Fig. 3.

The cut along the Doppler axis is obtained by setting $\tau = 0$ in (2) which yields

$$\chi_T(0,\nu) = \frac{1}{T}\int_0^T |u(t)|^2 \exp(j2\pi\nu t)\, dt. \tag{23}$$

Since only phase modulation is used, we have from (9) and (10) that

$$|u(t)| = 1. \tag{24}$$

That makes (23) independent of the detailed sequence of the modulation, yielding

$$|\chi_T(0,\nu)| = \left|\frac{\sin(\pi\nu T)}{\pi\nu T}\right|. \tag{25}$$

Using (25) in (8) we obtain

$$|\chi_{NT}(0,\nu)| = \left|\frac{\sin(\pi\nu NT)}{\pi\nu NT}\right| \tag{26}$$

A plot of (26) is given in Fig. 4. Using the periodicity (1), it can easily be shown that for any integer n

$$|\chi_{NT}(nT,\nu)| = |\chi_{NT}(0,\nu)|. \tag{27}$$

A periodicity of T is also maintained anywhere on the cuts $\nu = m/T$ $m = 0,\pm1,\pm2,\ldots$, which means that

$$\left|\chi_{NT}\left(\tau,\frac{m}{T}\right)\right| = \left|\chi_{NT}\left(\tau+nT,\frac{m}{T}\right)\right|. \tag{28}$$

However, off these cuts, the periodicity in general is of $2T$, namely, for any integer n

$$|\chi_{NT}(\tau,\nu)| = |\chi_{NT}(\tau+n2T,\nu)|. \tag{29}$$

Note that (25)–(29) apply to all periodic signals with angle modulation, and not only to those yielding perfect periodic autocorrelation. Another general result, which applies to all periodic signals, can be obtained directly from (8). It says that for any integer n

$$\left|\chi_{NT}\left(\tau,\frac{n}{T}\right)\right| = \left|\chi_{T}\left(\tau,\frac{n}{T}\right)\right|. \tag{30}$$

We have thus established that phase modulated periodic signals, which exhibit perfect periodic autocorrelation, have universal cuts of their PAF, along the delay and Doppler axes. These cuts are functions of only three parameters: the period, T, the number of phase modulation bits in the sequence, M, and the number of periods used in the correlation receiver, N.

The nature of the $\sin(Nx)/N\sin(x)$ function in (8) implies that for large N, the PAF will be quenched everywhere except for narrow strips parallel to the delay axis at $\nu = n/T$, $n = 0,\pm1,\pm2,\ldots$. Furthermore, from (26) and (27) we have that for all but $n = 0$, these strips must get a zero value at $\tau = kT$, $k = 0,\pm1,\pm2,\ldots$.

The behavior of the PAF off the two major axes, must be a function of the detailed sequence of the signal. A few examples are now discussed.

V. DETAILED EXAMPLES

Detailed results of the PAFs are given in this section for the signals corresponding to the sequences

$$U_1 = \{1\ 1\ 1\ \beta_1\ \beta_1\ 1\ \beta_1\}$$

$$U_2 = \{1\ 1\ 1\ \beta_2\ \beta_2\ \beta_2\ 1\ \beta_2\ \beta_2\ 1\ \beta_2\}$$

where

$$\beta_1 = \exp[j\cos^{-1}(-3/4)]$$

$$\beta_2 = \exp[j\cos^{-1}(-5/6)].$$

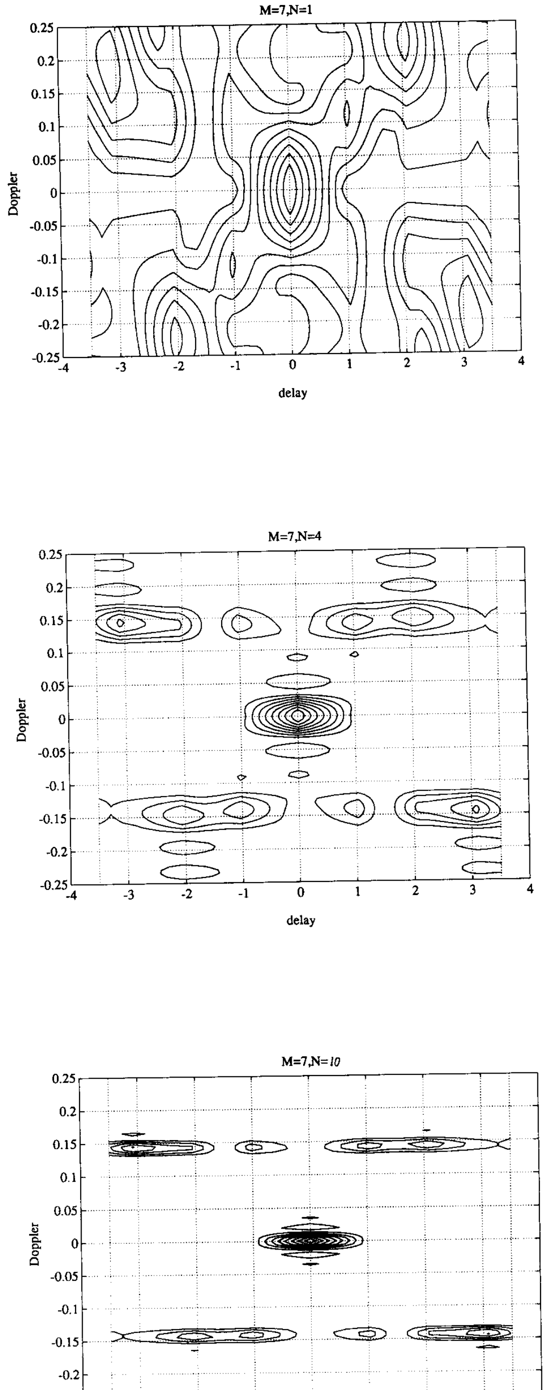
Starting with the first signal, in which the number of bits is $M = 7$, Figs. 5(a) to 5(c) present contour plots of the absolute value of the ambiguity function, for the three cases $N = 1$, 4, and 10. The scales are normalized with respect to the bit duration t_b. Namely, the delay axis is of τ/t_b, and the Doppler axis is of νt_b. Since the signal period is $T = Mt_b$, the ambiguity function repeats itself every $2M$ normalized delay units. (The cuts at $\nu = n/M$ repeat every M normalized delay units. The pronounced strips, parallel to the delay axis, appear at normalized Doppler of n/M. There are contour lines at values $0.11, 0.22, \ldots, 0.77$, and 0.88.

The corresponding 3-D plots are given in Figs. 6(a) to 6(c). The prominent feature of the ambiguity function, when $N > 1$, is the quenching of the function except at the strips $\nu t_b = n/M$. With the strips getting narrower as N increases. The cuts of the PAF at $\nu t_b = n/M$ are independent of the number of periods N. Several cuts, for different values of n, are given in Fig. 7. Note how the peak of the cuts moves away from $\tau = 0$ as n increases above zero. This phenomena is found also in sinusoidal, triangular, and other periodic frequency modulation of CW signals [4].

The ambiguity function of the signal U_2, in which $M = 11$, is analyzed in a similar way in Figs. 8–10. A demonstration of the periodicity of $2Mt_b$ is given (for the signal U_1) in Fig. 11.

VI. CONCLUSIONS

The CW signals discussed are unique in producing perfect periodic autocorrelation. In order to study their application as radar signals, it was necessary to observe the response of a correlation receiver to a Doppler-shifted version of the transmitted signal. This response depends on the duration of the reference signal in the receiver. An appropriate PAF was defined which represents that response when the reference signal is constructed from an integral number of periods of the transmitted signal. Several general features of this PAF were discussed, as well as specific features, typical to the special signals with perfect periodic autocorrelation.



(a)

(b)

(c)

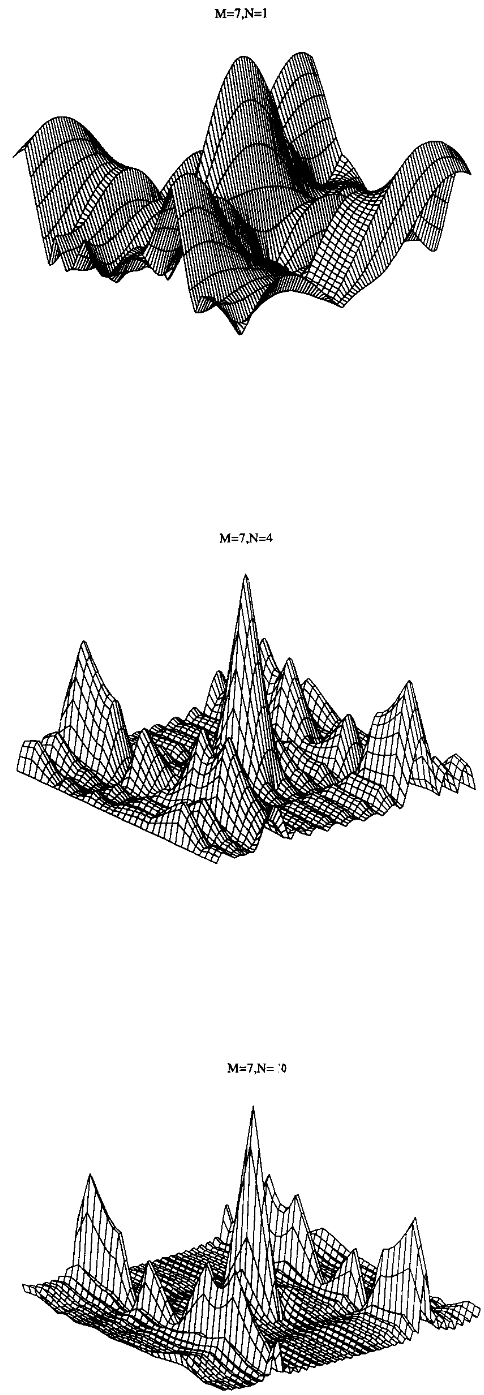


Fig. 5. Contour plot of PAF of signal U_1 when $N = 1$, 4, and 10.

Fig. 6. 3-D view of ambiguity function of signal U_1 when $N = 1$, 4, and 10.

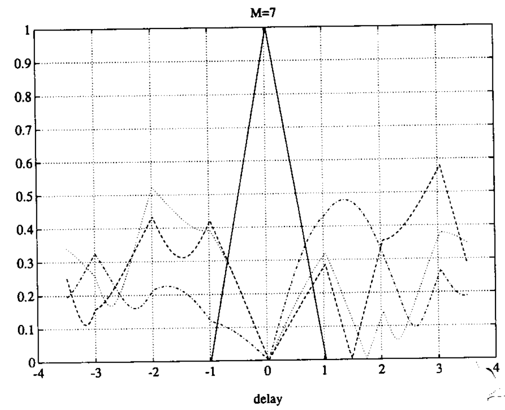


Fig. 7. Cuts of ambiguity function of U_1 along $\nu M t_b = 0$ (solid), 1 (dash), 2 (dots), and 3 (dash-dot).

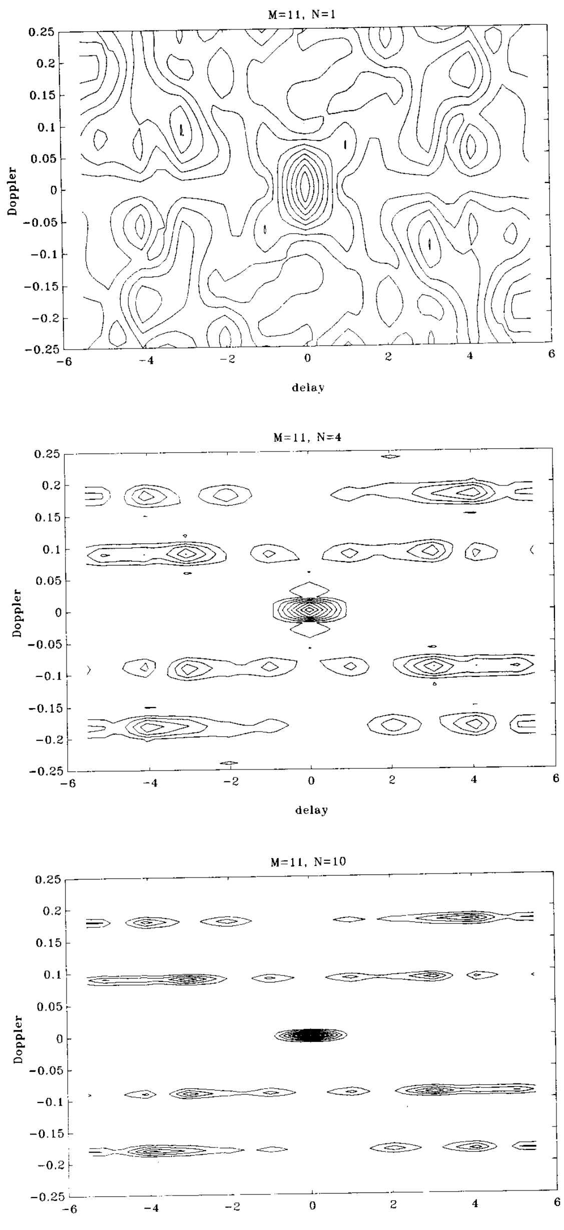


Fig. 8. Contour plot of periodic ambiguity function of signal U_2 when $N = 1$, 4, and 10.

(a)

(b)

(c)

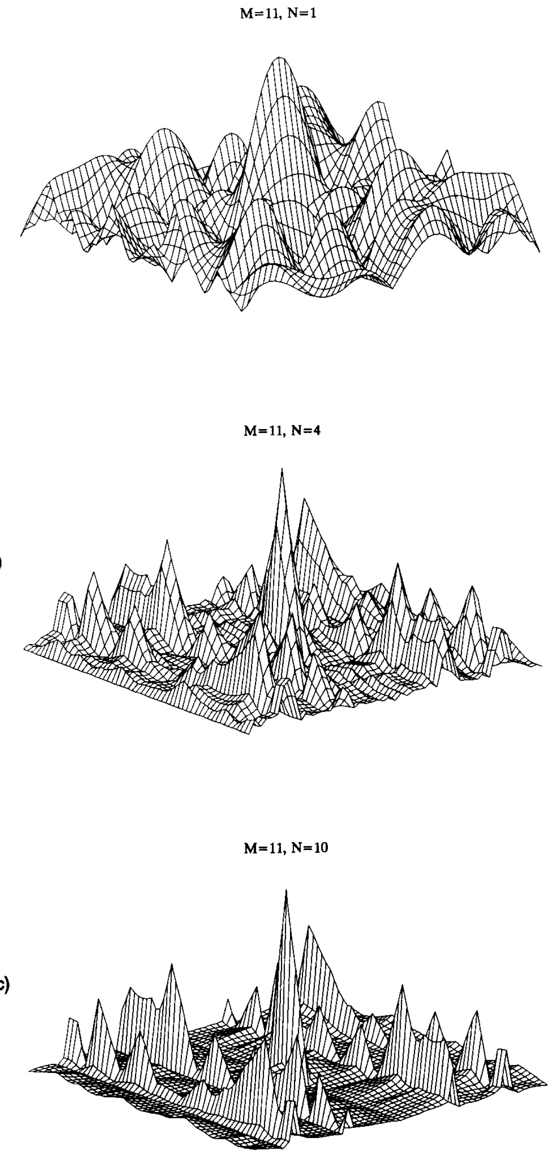


Fig. 9. 3-D view of ambiguity function of signal U_2 when $N = 1$, 4, and 10.

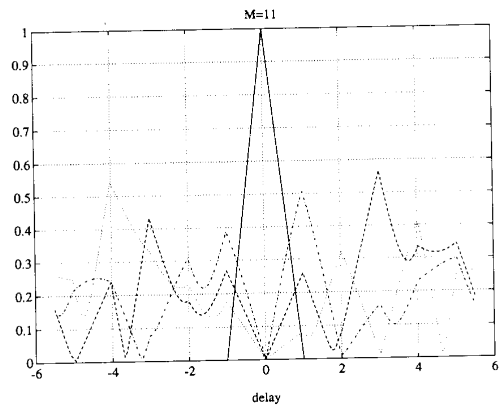


Fig. 10. Cuts of ambiguity function of U_2 along $\nu M t_b = 0$ (solid), 1 (dash), 2 (dots), and 3 (dash-dot).

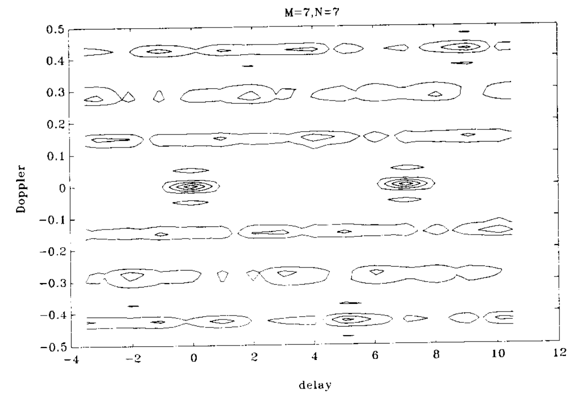


Fig. 11. Contour plot of PAF of signal U_1 when $N = 7$. Horizontal scale stretches over $2Mt_b$ in order to demonstrate that this is the periodicity of the ambiguity function.

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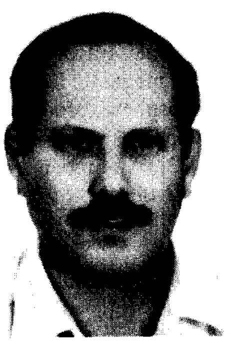

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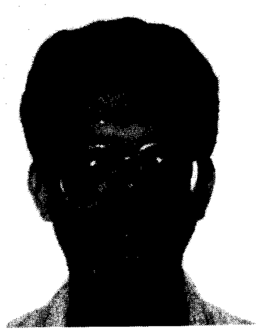

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