

Lowest GDOP in 2-D scenarios

N.Levanon

Abstract: Position determination in two-dimensional (2-D) scenarios is examined. The results can apply to position location in cellular systems using the system infrastructure. It is shown that the lowest possible geometric dilution of precision (GDOP) attainable from range or pseudo-range measurements to N optimally located points is $2/\sqrt{N}$. The significance of $2/\sqrt{N}$ in bearing-only position determination is also pointed out. Contour GDOP maps demonstrate that it is worthwhile to add one absolute-range measurement (more difficult to implement) to the set of pseudo-ranges (relatively easy to implement).

1 Introduction

GDOP describes the effect of geometry on the relationship between measurement error and position determination error [1, 2]. GDOP is a dimensionless expression if all the measurements use the same unit as the position unit (e.g. metres). GDOP expression can be relatively simple if all the measurements exhibit the same RMS error.

Some important positioning applications can be described (or approximated) as 2-D scenarios, in which the sensors and the unknown target position are practically on a single plane. One example is positioning in cellular-phone systems, using only measurements with respect to the terrestrial base stations.

The available measurements could be passive (one-way), active (two-way) or a combination of both. An example of passive measurements is the case in which a signal originating at a single unknown location (e.g. the phone) at an unknown time is received by several synchronised sensors at different known locations (e.g. the base stations). Another example is the reciprocal case, in which signals originating simultaneously at several known locations (e.g. base stations of a CDMA system) are received by a single receiver (e.g. the phone) at the unknown location. Passive measurements yield time of arrival (TOA) or time difference of arrival (TDOA) measurements (relative delays) that can be converted to range-differences or pseudo-ranges. Active measurements involve signals travelling in both directions, either through a response (in communications) or through reflection (radar). Active measurements usually produce a round-trip delay (RTD) that yields absolute-range.

When designing such a positioning system, it is useful to know what is the relative contribution to accuracy of the various types of measurement and what is the best (lowest) attainable GDOP. The best GDOP occurs at the most favourable location relative to optimally situated sensors. Intuition suggests that optimum distribution of N sensors is

at the vertices of an N -sided regular polygon and that the best GDOP will be found at the centre of the polygon.

We will begin by defining the two kinds of measurement (pseudo-range and absolute-range) and the corresponding GDOP. We will then show analytically that, at the centre of an N -sided regular polygon, $N \geq 3$, the GDOP is equal to $2/\sqrt{N}$, no matter whether the measurements are ranges or pseudo-ranges to the N vertices of the polygon. (Lee [1] reached the result of $2/\sqrt{N}$ with respect to pseudo-ranges only.)

We then consider GDOP contour maps of an extended area around the polygon. For pseudo-ranges, the maps show that the GDOP increases monotonically and rather rapidly outside the polygon. For absolute-ranges, the maps show that the GDOP declines back to its minimum value of $2/\sqrt{N}$ outside the polygon edges, before rising again with distance, at a much lower rate than observed in the pseudo-ranges case. The special case of $N=2$ is discussed, and we then demonstrate similar behaviour for positioning based on bearing measurements.

2 Measurements, errors and GDOP

Let the range from the target to the i th sensor be given by

$$R_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + z_i^2} \quad i = 1, \dots, N \quad (1)$$

where x_i, y_i, z_i are the known co-ordinates of the i th sensor, and x, y are the unknown co-ordinates of the target. Note that eqn. 1 allows for height variations of the sensors relative to the horizontal plan ($z=0$) passing through the target.

Absolute-range is a measurement of R_i that can be obtained from RTD measurement. RTD requires two-way signals between the target and the sensor. Obtaining such a measurement involves an active, well-calibrated transponder or reflection.

Range-difference is obtained from subtracting two TOA measurements

$$\Delta R_i = R_i - R_1 \quad i = 2, \dots, N \quad (2)$$

The errors of the several ΔR_i are not independent of each other, as the error in the arrival time of the signal travelling over R_1 affects all the measurements. However, the

measurements can be made independent by using the pseudo-range concept

$$PR_i = R_i + \rho \quad i = 1, \dots, N \quad (3)$$

where ρ is an arbitrary range offset, common to all N pseudo-range measurements. ρ can have a physical meaning, such as the delay of the time reference (times the velocity of propagation) with respect to which all the relative delay measurements were made.

Estimating the target co-ordinates can be done by either (a) using $N - 1$ range-difference 'measurements', whose errors are dependent, and solving for two unknowns $[x \ y]$, or (b) using N pseudo-range measurements, whose errors are independent, and solving for three unknowns $[x \ y \ \rho]$.

Both approaches will yield exactly the same result if the dependency between the measurement errors in approach (a) is properly handled.

The range (delay) measurements suffer from thermal, instrument and propagation errors. The total error cannot be simply modelled. However, as we are interested in comparing the geometrical effect on the positioning accuracy, we will make the over-simplified assumption that the errors in all actual measurements (pseudo-range or absolute-range) are random, independent, have zero mean and have an identical RMS value σ_R .

This simplifying assumption implies, for example, that, for the case of four pseudo-range measurements, the vector of measurements is

$$\mathbf{M} = [PR_1 PR_2 PR_3 PR_4]^T \quad (4)$$

The vector of unknowns is

$$\boldsymbol{\theta} = [x \ y \ \rho]^T \quad (5)$$

and the error covariance matrix is

$$\mathbf{Q} = \sigma_R^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6)$$

On the other hand, the same set of measurements can be used as three range-difference measurements with the following new representation:

$$\mathbf{M}_1 = [\Delta R_2 \ \Delta R_3 \ \Delta R_4]^T \quad (7)$$

The vector of unknowns is

$$\boldsymbol{\theta}_1 = [x \ y]^T \quad (8)$$

and the error covariance matrix is

$$\mathbf{Q}_1 = \sigma_R^2 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad (9)$$

The terms '2' in the new error covariance matrix reflect the fact that a range-difference measurement constitutes the differences between two range (relative delay) measurements. The off-diagonal elements '1' reflect the fact that range-difference 'measurements' are not independent, as a common delay measurement is used to generate each pair of range-differences.

The best, linear, unbiased estimate (BLUE) [3] of the target's two position unknowns, yields variance given by

$$\text{Var } \hat{\boldsymbol{\theta}} = \text{diag}(\mathbf{H}^T \mathbf{Q}^{-1} \mathbf{H})^{-1} \quad (10)$$

where \mathbf{H} is the matrix of partial derivatives of the noise-free measurement equations with respect to the unknown

parameters (Jacobian). For example, for the pseudo-range measurements approach (four measurements),

$$\mathbf{H} = \begin{bmatrix} \frac{\partial PR_1}{\partial x} & \frac{\partial PR_1}{\partial y} & \frac{\partial PR_1}{\partial \rho} \\ \frac{\partial PR_2}{\partial x} & \frac{\partial PR_2}{\partial y} & \frac{\partial PR_2}{\partial \rho} \\ \frac{\partial PR_3}{\partial x} & \frac{\partial PR_3}{\partial y} & \frac{\partial PR_3}{\partial \rho} \\ \frac{\partial PR_4}{\partial x} & \frac{\partial PR_4}{\partial y} & \frac{\partial PR_4}{\partial \rho} \end{bmatrix} \quad (11)$$

and, for the range-difference approach (three range-differences),

$$\mathbf{H}_1 = \begin{bmatrix} \frac{\partial \Delta R_2}{\partial x} & \frac{\partial \Delta R_2}{\partial y} \\ \frac{\partial \Delta R_3}{\partial x} & \frac{\partial \Delta R_3}{\partial y} \\ \frac{\partial \Delta R_4}{\partial x} & \frac{\partial \Delta R_4}{\partial y} \end{bmatrix} \quad (12)$$

In 2-D cases the GDOP will be defined as in [1, 2], namely

$$GDOP = \frac{\sqrt{\text{Var } x + \text{Var } y}}{\sigma_R} \quad (13)$$

where σ_R is either the RMS error common to all the measurements, or the square root of the mean square of the different RMS errors. Note that, in 3-D situations, the acronym HDOP is used to express the right-hand side expression of eqn. 13.

When the error covariance matrix is not diagonal, as in eqn. 9, the GDOP will be given by

$$GDOP = \sqrt{\mathbf{G}_{1,1} + \mathbf{G}_{2,2}} \quad (14)$$

where

$$\mathbf{G} = \frac{1}{\sigma_R^2} (\mathbf{H}_1^T \mathbf{Q}_1^{-1} \mathbf{H}_1)^{-1} \quad (15)$$

When using N pseudo-range measurements, or N absolute-range measurements, $\mathbf{Q} = \sigma_R^2 \mathbf{I}$, where \mathbf{I} is an $N \times N$ identity matrix. Hence,

$$\text{Var } \hat{\boldsymbol{\theta}} = \sigma_R^2 \text{diag}(\mathbf{H}^T \mathbf{H})^{-1} \quad (16)$$

and the GDOP eqn. 14 will still apply if we define

$$\mathbf{G} = (\mathbf{H}^T \mathbf{H})^{-1} \quad (17)$$

The partial derivatives required in \mathbf{H} , of pseudo-range, range and range-difference measurements are, respectively,

$$\frac{\partial PR_i}{\partial x} = \frac{x - x_i}{R_i}; \quad \frac{\partial PR_i}{\partial y} = \frac{y - y_i}{R_i}; \quad \frac{\partial PR_i}{\partial \rho} = 1 \quad (18)$$

$$\frac{\partial R_1}{\partial x} = \frac{x - x_1}{R_1}; \quad \frac{\partial R_1}{\partial y} = \frac{y - y_1}{R_1}; \quad \frac{\partial R_1}{\partial \rho} = 0 \quad (19)$$

$$\frac{\partial \Delta R_i}{\partial x} = \frac{\partial R_i}{\partial x} - \frac{\partial R_1}{\partial x}; \quad \frac{\partial \Delta R_i}{\partial y} = \frac{\partial R_i}{\partial y} - \frac{\partial R_1}{\partial y} \quad (20)$$

For example, when solving for the target position using N absolute-range measurements, the vector of unknowns is as in eqn. 8. Using eqn. 19, the partial derivative matrix

appears in eqn. 21 and is used in eqns. 17 and 14 to obtain the GDOP.

$$\mathbf{H} = \begin{bmatrix} \frac{x-x_1}{R_1} & \frac{y-y_1}{R_1} \\ \frac{x-x_2}{R_2} & \frac{y-y_2}{R_2} \\ \dots & \dots \\ \frac{x-x_N}{R_N} & \frac{y-y_N}{R_N} \end{bmatrix} \quad (21)$$

3 GDOP at centre of regular polygon

We will now develop the two-dimensional GDOP at the centre of an N -sided regular polygon, $N \geq 3$, when the available measurements are either ranges or pseudo-ranges to N sensors located at the vertices of the polygon.

For a polygon as defined in Fig. 1, the co-ordinates of the n th sensor are

$$x_n = a \cos \frac{2\pi(n-1)}{N}, \quad y_n = a \sin \frac{2\pi(n-1)}{N} \quad n = 1, 2, \dots, N \quad (22)$$

The corresponding ranges to a target located at (x, y) are

$$r_n = \sqrt{(x-x_n)^2 + (y-y_n)^2} \quad (23)$$

The partial derivatives are

$$\frac{\partial r_n}{\partial x} = \frac{x-x_n}{r_n}, \quad \frac{\partial r_n}{\partial y} = \frac{y-y_n}{r_n} \quad (24)$$

At the centre of the polygon, namely at $(x=0, y=0)$,

$$\frac{\partial r_n}{\partial x} = -\cos \frac{2\pi(n-1)}{N}, \quad \frac{\partial r_n}{\partial y} = -\sin \frac{2\pi(n-1)}{N}, \quad \text{at } x=0, y=0 \quad (25)$$

When the available measurements are absolute-ranges then the matrix of partial derivatives becomes

$$\mathbf{H} = - \begin{bmatrix} \cos \frac{2\pi 0}{N} & \sin \frac{2\pi 0}{N} \\ \vdots & \vdots \\ \cos \frac{2\pi(N-1)}{N} & \sin \frac{2\pi(N-1)}{N} \end{bmatrix} \quad (26)$$

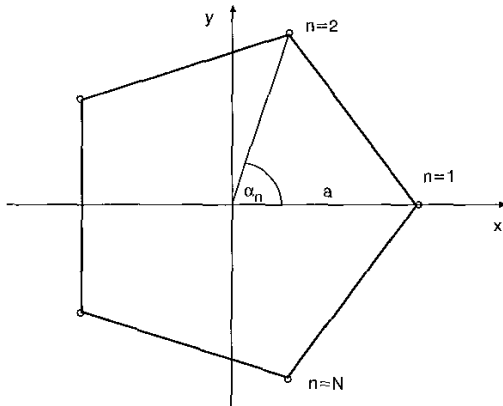


Fig. 1 Five-sided regular polygon

Using the Fourier summation formulas,

$$\sum_{n=0}^{N-1} \cos^2(2\pi n/N) = \frac{N}{2} \quad N \geq 3 \quad (27)$$

$$\sum_{n=0}^{N-1} \sin^2(2\pi n/N) = \frac{N}{2} \quad N \geq 3 \quad (28)$$

$$\sum_{n=0}^{N-1} \cos(2\pi n/N) \sin(2\pi n/N) = 0 \quad (29)$$

and eqns. 13, 14 and 17 yield

$$\mathbf{G} = (\mathbf{H}^T \mathbf{H})^{-1} = \begin{bmatrix} \frac{2}{N} & 0 \\ 0 & \frac{2}{N} \end{bmatrix} \quad (30)$$

$$GDOP = \sqrt{\mathbf{G}_{1,1} + \mathbf{G}_{2,2}} = \frac{2}{\sqrt{N}} \quad (31)$$

If the available measurements were pseudo-ranges,

$$PR_n = \sqrt{(x-x_n)^2 + (y-y_n)^2} + \rho \quad (32)$$

then the matrix of partial derivatives at the origin becomes an $N \times 3$ matrix

$$\mathbf{H} = \begin{bmatrix} -\cos \frac{2\pi 0}{N} & -\sin \frac{2\pi 0}{N} & 1 \\ \vdots & \vdots & \vdots \\ -\cos \frac{2\pi(N-1)}{N} & -\sin \frac{2\pi(N-1)}{N} & 1 \end{bmatrix} \quad (33)$$

Using the summation formulas eqns. 27–29 and

$$\sum_{n=0}^{N-1} \cos(2\pi n/N) = 0, \quad \sum_{n=0}^{N-1} \sin(2\pi n/N) = 0 \quad (34)$$

yields

$$\mathbf{G} = (\mathbf{H}^T \mathbf{H})^{-1} = \begin{bmatrix} \frac{2}{N} & 0 & 0 \\ 0 & \frac{2}{N} & 0 \\ 0 & 0 & \frac{1}{N} \end{bmatrix} \quad (35)$$

Hence,

$$GDOP = \sqrt{\mathbf{G}_{1,1} + \mathbf{G}_{2,2}} = \frac{2}{\sqrt{N}} \quad (36)$$

Note that, for pseudo-range measurements only, the result of eqn. 36 appeared in [1] as an exact result for $N=3$ and as an approximate result for $N>3$.

Comparing eqn. 36 with eqn. 31 indicates that, at the centre of the polygon, the GDOP is identical for absolute-range and pseudo-range measurements. This is confirmed by the GDOP contour maps in the following Section. However, the maps will show that, away from the polygon centre, the GDOP contours behave differently for the two types of measurement.

4 GDOP contour maps

Figs. 2–7 present GDOP contour maps for $N=3, 4$ and 5 . For each N , there are two maps, one for absolute-range measurements and the other for pseudo-range measurements.

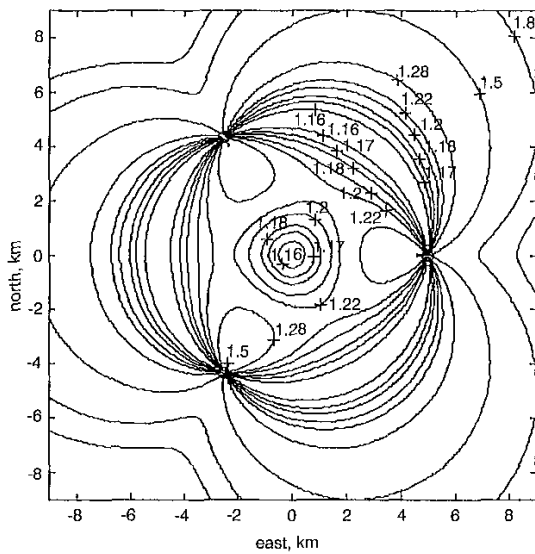


Fig. 2 GDOP contours using three absolute-range measurements

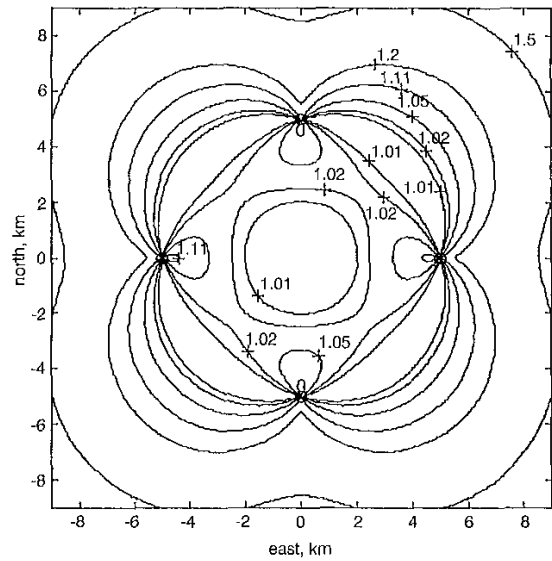


Fig. 4 GDOP contours using four absolute-range measurements

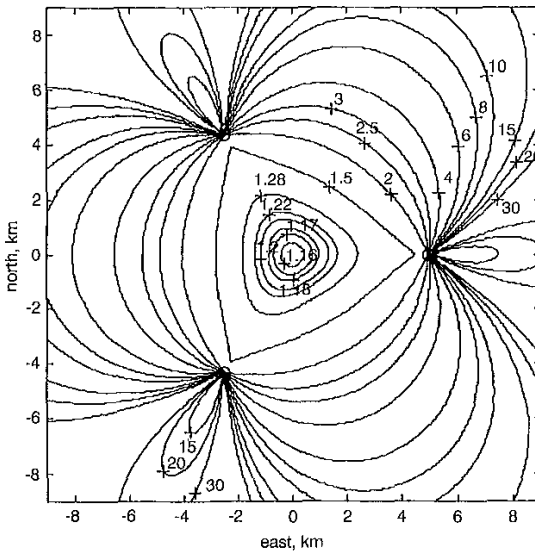


Fig. 3 GDOP contours using three pseudo-range measurements

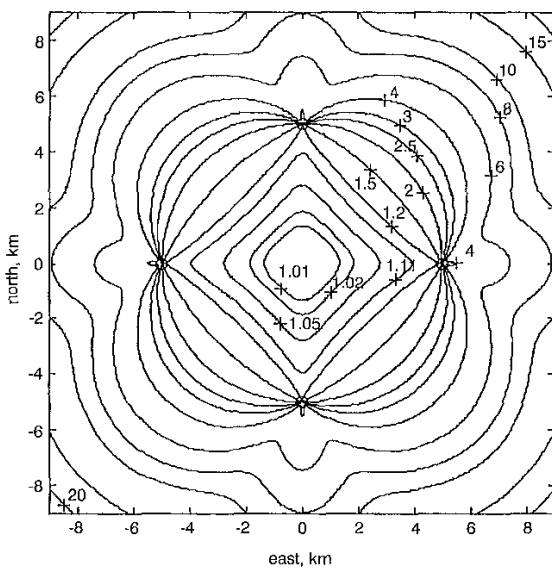


Fig. 5 GDOP contours using four pseudo-range measurements

In all the Figures, note the good agreement between the lowest contour value and the theoretical lower GDOP limit of $2/\sqrt{N}$ (see Table 1).

Beginning with Fig. 2, which applies to range measurements, note that the lowest contour ($=1.16$) reappears outside each side of the polygon (triangle). It can be shown that the two minima are located on the normal bisector of the side of the triangle and, for $N=3$ only, at equal but opposite distances from the side. Fig. 3, which applies to pseudo-range measurements, exhibits only one mini-

num, at the centre of the polygon. The other major difference is the rapid increase of the GDOP with distance, outside the polygon. Similar behaviour can be observed in the other two pairs. Figs. 4 and 5 for $N=4$, and Figs. 6 and 7 for $N=5$.

Finally, it is interesting to note the effect of mixed measurements. Intuitively, it is easy to see that, if there are N pseudo-ranges to the N vertices of an N -sided polygon, then adding a single absolute-range measurement to one of the vertices has the effect of nearly converting all the pseudo-range measurements to absolute-range measurements. This phenomenon is demonstrated in Fig. 8, for the case of a four-sided polygon. Comparing Fig. 8 with Figs. 4 and 5, note that, near the outer edges, the contour level is $\text{GDOP} = 1.9$ in Fig. 8, which is closer to $\text{GDOP} = 1.5$ (Fig. 4) than to $\text{GDOP} = 15$ (Fig. 5). Note also from Fig. 8 that, as it was generated using five measurements, the lowest GDOP contour is 0.92 (the actual GDOP minimum in Fig.

Table 1: Agreement between lowest contour value and $2/\sqrt{N}$

N	3	4	5
$2/\sqrt{N}$	1.1547	1	0.8944
Lowest contour	1.16	1.01	0.90

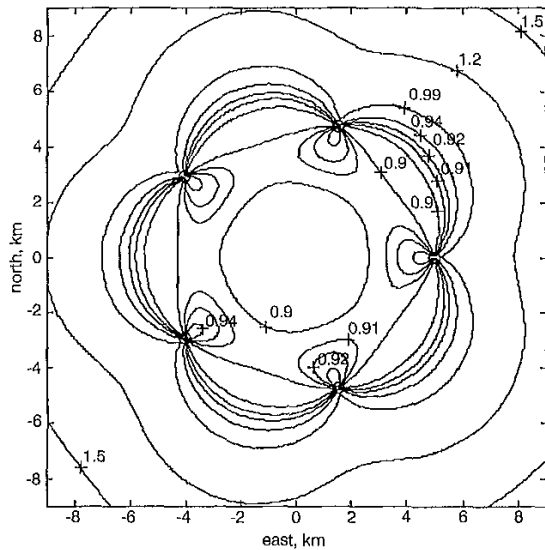


Fig. 6 GDOP contours using five absolute-range measurements

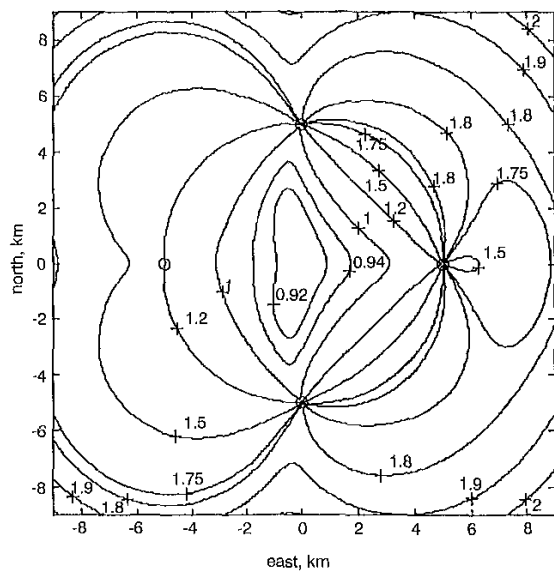


Fig. 8 GDOP contours using four pseudo-ranges plus one absolute-range (to the rightmost vertex)

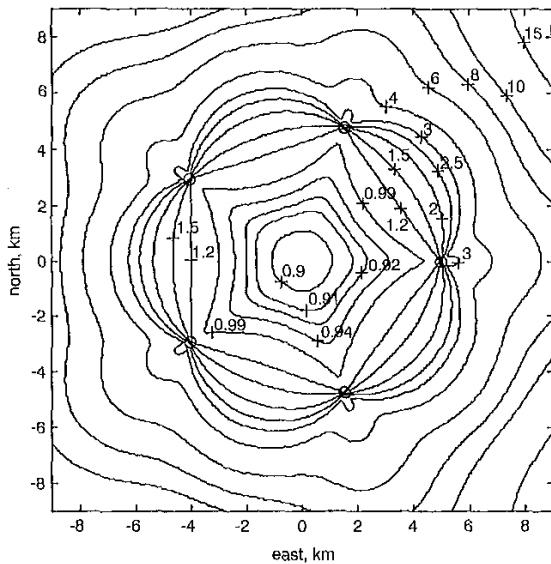


Fig. 7 GDOP contours using five pseudo-range measurements

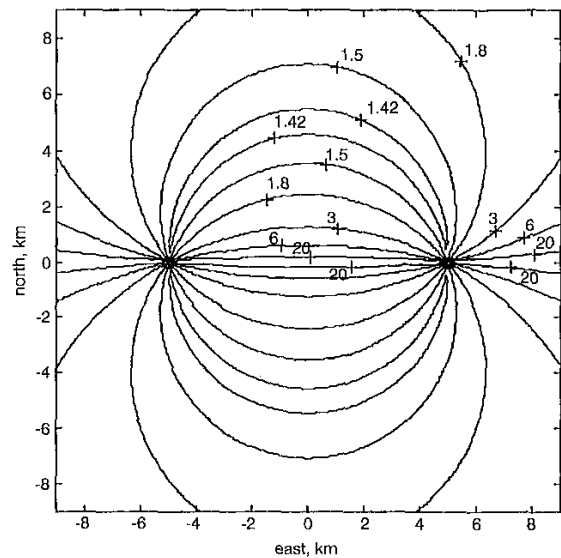


Fig. 9 GDOP contours using two absolute-range measurements

8 was 0.9119), which is closer to $2/\sqrt{5} = 0.8944$ than to $2/\sqrt{4} = 1$.

5 Special case $N=2$

The $N=2$ case is special for two reasons: first, two pseudo-range measurements are not enough to solve for a position, and, secondly, absolute-range measurements to two points yield two solutions, symmetrical with respect to the baseline connecting the two points. Fig. 9 demonstrates that the lowest GDOP still obeys the rule of $2/\sqrt{N} = 1.414$.

Fig. 9 was obtained using the partial derivative matrix

$$H = \begin{bmatrix} \frac{x-x_1}{R_1} & \frac{y-y_1}{R_1} \\ \frac{x-x_2}{R_2} & \frac{y-y_2}{R_2} \end{bmatrix} \quad (37)$$

and the error covariance matrix

$$Q = \sigma_R^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (38)$$

Although two pseudo-range measurements are not enough for a solution, the combination of two pseudo-range measurements and one absolute-range measurement ($R_1, R_1 + \rho, R_2 + \rho$) does yield a solution (albeit with ambiguity). The GDOP for that scenario can be obtained using the same H as in eqn. 37, but modifying Q to

$$Q = \sigma_R^2 \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \quad (39)$$

Another sufficient combination is one absolute-range and one range-sum measurement ($R_1, R_1 + R_2$). The GDOP for

that scenario can be obtained using the same H as in eqn. 37, but modifying Q to

$$Q = \sigma_R^2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad (40)$$

The lowest GDOP for the last two combinations of measurements are 1.94 and 1.62, respectively.

6 Bearing-only position determination

In bearing-only position determination [4], each sensor measures the angle α to the target, relative to the x -axis. Let us assume that all angle measurement errors are IID with RMS of σ_α . As α is given by

$$\alpha_i = \tan^{-1} \left(\frac{y - y_i}{x - x_i} \right) \quad (41)$$

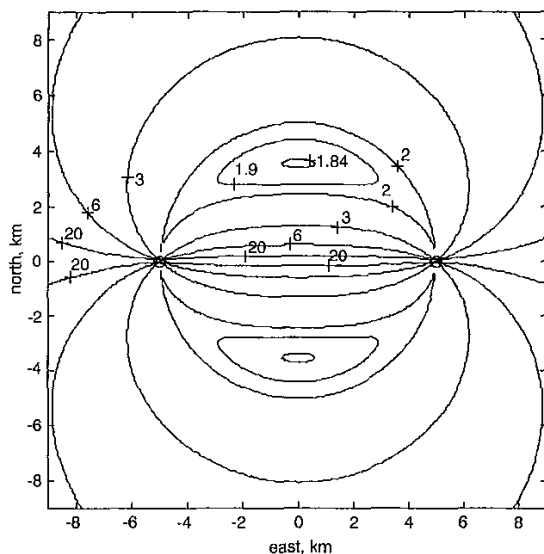


Fig. 10 Normalised GDOP contours using two bearing measurements
Position error/radius (for $\sigma_\alpha = 1$ rad), using bearing

the partial derivatives with respect to the target co-ordinates are

$$\frac{\partial \alpha_i}{\partial x} = -\frac{y - y_i}{R_i^2}; \quad \frac{\partial \alpha_i}{\partial y} = \frac{x - x_i}{R_i^2} \quad (42)$$

The partial derivative matrix becomes

$$H = \begin{bmatrix} \frac{\partial \alpha_1}{\partial x} & \frac{\partial \alpha_1}{\partial y} \\ \vdots & \vdots \\ \frac{\partial \alpha_N}{\partial x} & \frac{\partial \alpha_N}{\partial y} \end{bmatrix} \quad (43)$$

In this case, the measurement errors are in units of radians, and the position error is in units of metres. To obtain a dimensionless GDOP, we will define a normalised GDOP.

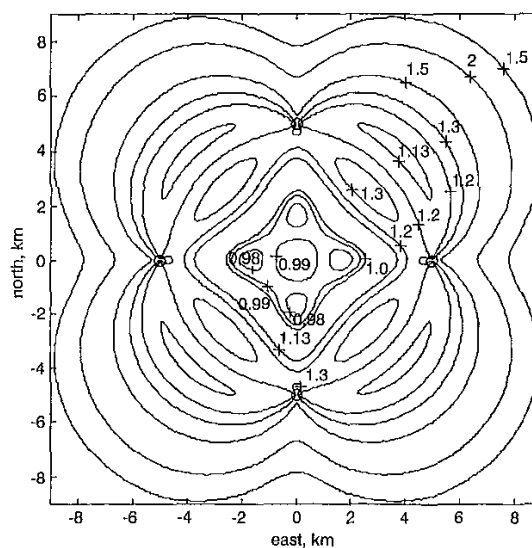


Fig. 12 Normalised GDOP contours using four bearing measurements
Position error/radius (for $\sigma_\alpha = 1$ rad), using bearing

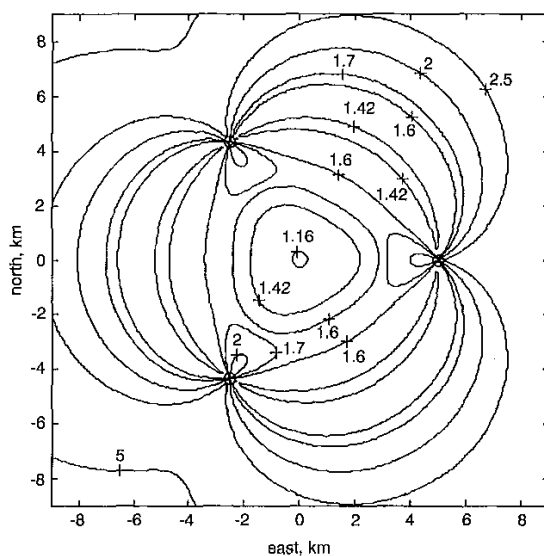


Fig. 11 Normalised GDOP contours using three bearing measurements
Position error/radius (for $\sigma_\alpha = 1$ rad), using bearing

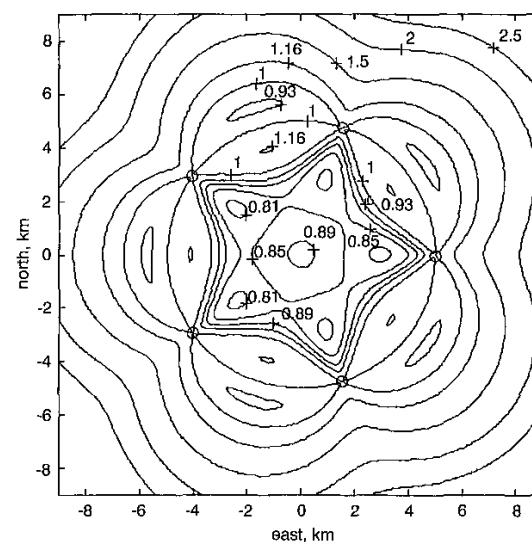


Fig. 13 Normalised GDOP contours using five bearing measurements
Position error/radius (for $\sigma_\alpha = 1$ rad), using bearing

For the case where the N sensors are located at the vertices of a regular N -sided polygon, we will define the GDOP as

$$GDOP = \frac{1}{a} \sqrt{\mathbf{G}_{1,1} + \mathbf{G}_{2,2}} \quad (44)$$

where a is the distance from a vertex to the centre of the polygon (see Fig. 1), and

$$\mathbf{G} = (\mathbf{H}^T \mathbf{H})^{-1} \quad (45)$$

With this definition, conversion from the normalised GDOP to the position RMS error is through the product $a\sigma_\alpha GDOP$, where σ_α is the angle measurement RMS error in radians.

Contour plots of the normalized GDOP are presented in Figs. 10–13, for the cases $N=2, \dots, 5$, respectively. The contour maps reveal an interesting result: at the centre of an N -sided regular polygon ($N > 2$), the normalised GDOP is equal to $2/\sqrt{N}$, as in the case of range or pseudo-range measurements. However, the contour maps also show that, for $N > 3$, the global minima are slightly lower than $2/\sqrt{N}$ and are located symmetrically off the centre.

7 Conclusions

We showed that, in 2-D scenarios, for position determination, based on N measurements ($N > 2$), the lowest possible GDOP is $2/\sqrt{N}$, no matter whether the measurements are passive (yielding pseudo-ranges) or active (yielding absolute-ranges). This lowest value will occur when the range or pseudo-range measurements are with respect to N points located at the vertices of a regular N -sided polygon, and

will be found at the most favourable location, which is the centre of the polygon.

The difference between the two types of measurement is prominent in the rate at which the GDOP increases away from the centre, mainly outside the polygon. Pseudo-range measurements yield a much larger rate of GDOP increase. Hence, when the target is outside the polygon, it is advantageous to use absolute-range measurements, if available. However, most of the advantages of using absolute-range rather than pseudo-range can be gained by using at least one absolute-range measurement in addition to pseudo-range measurements. In cellular phone systems, it is practical to obtain one absolute-range using RTD measurement with respect to the serving base station.

When $N=2$, the lowest GDOP bound of $2/\sqrt{N}$ still holds for absolute-range measurements. However, two ambiguous solutions exist. For bearing-only measurements, a normalised GDOP value of $2/\sqrt{N}$ is still found at the centre of the polygon, but it is not necessarily the lowest.

8 References

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