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Range sidelobes blanking by comparing outputs of contrasting mismatched filters

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Abstract: Range sidelobes, a major shortcoming of radar pulse compression, are often reduced through the use of mismatched filters. The authors propose to blank the remaining sidelobes by using two or more mismatched filters, whose sidelobes are designed to peak at different delays. The authors show how to design such filters and present promising simulation results with two or more mismatched filters.

1 Introduction

In radar pulse-compression processing, the matched filter is often replaced by a mismatched filter because the latter exhibits lower delay-sidelobes. The sidelobe level (peak or integrated) usually drops as the filter's length increases, while the signal-to-noise-ratio (SNR) loss usually levels off [1]. An important remaining argument against a long mismatched filter is the extended delay span of its sidelobes, and the increasing likelihood to overlap neighbouring targets. This paper suggests a sidelobe-blanking processor that will help distinguish between sidelobes and neighbouring targets, thus facilitating the use of longer mismatched filters.

Sidelobe blanking is used extensively in radar antennas [2]. There two separate receiving channels are required, one connects to the high-gain narrow-beam antenna, and the other is connected to a low-gain wide-beam auxiliary antenna. Each receiver receives a different signal and different additive noise. Our proposed range sidelobe-blanking system bears some resemblance to the antenna sidelobe-blanking system, but operates on the same received signal plus noise. In our system, the split into two channels occurs after detection and M -pulse integration (if used).

2 Sidelobe-blanking concept

The blanking concept can be implemented by a receiver whose block diagram appears in Fig. 1. In coherent radar

the receiver band-pass filter is followed by synchronous detection, sampling and M -pulse integration. Not shown are the FFTs required to perform coherent summation of M -pulses at each range resolution bin, to account for the inter-pulse phase ramp introduced by the target's Doppler shift. We will assume that the output of the ' M -pulse integration' block is one output of the FFTs. The ensuing shift register contains the delay elements. Normally the (complex) values in these delay elements, which contain signal plus noise, are cross-correlated with one reference sequence (filter), matched or mismatched, to produce the pulse-compression output for that Doppler. The pulse-compression output is compared to a threshold, and targets are declared at those delays in which the output exceeds the threshold. Our sidelobe-blanking concept requires that the same signal plus noise values, stored in the delay register, will be also correlated with a second filter.

First, we should point out that this kind of parallel processing does not imply that the power is split between the two channels. Sidelobe blanking is expected to be performed in a digital signal processing block of the receiver, where the signal (plus noise) is a sequence of complex digital numbers, which can be correlated simultaneously, and with no loss, with any number of references.

In the processor, the magnitude or square output values of the two filters are compared. Only when they (u and v) have similar values, one of them is allowed to proceed to the threshold decision stage. The second filter should be

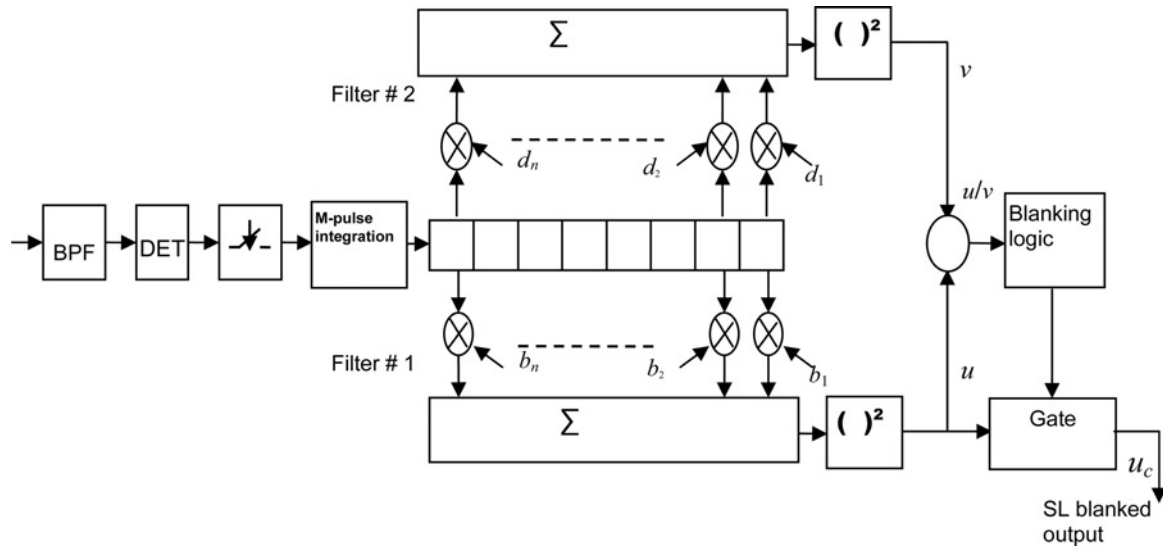


Figure 1 Conceptual block diagram of pulse-compression sidelobe blanking

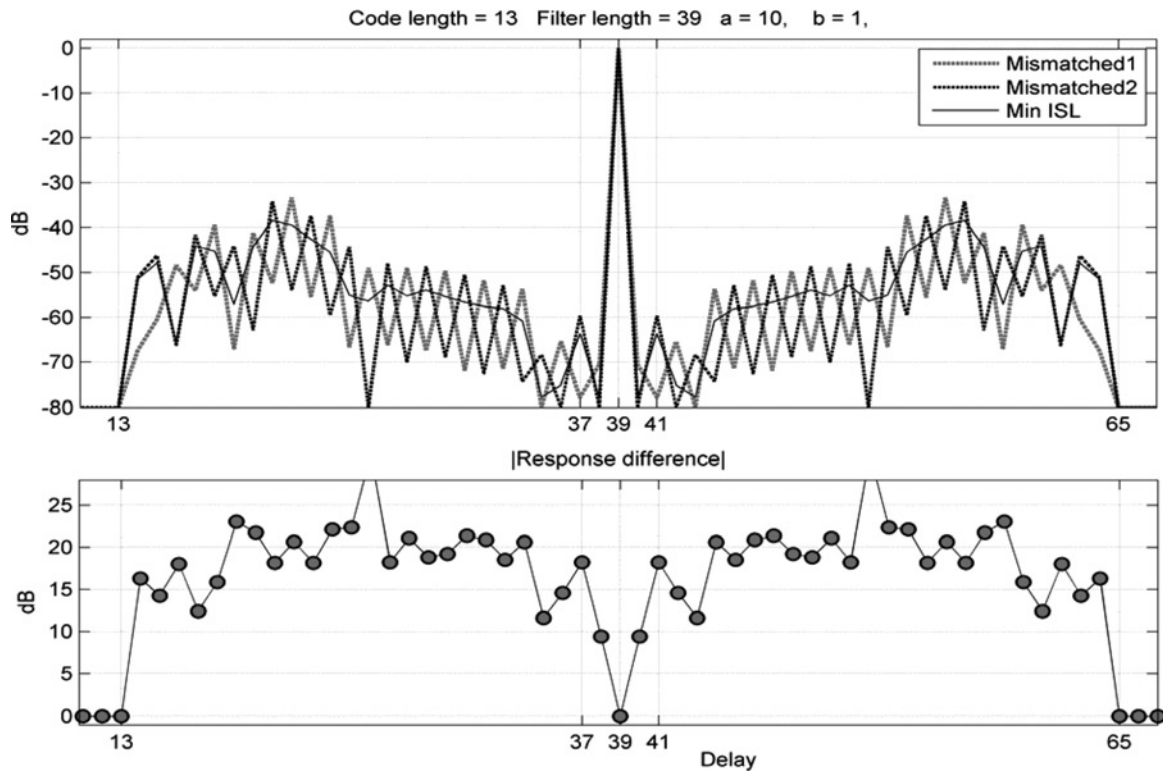


Figure 2 Responses of two contrasting mismatched filters of length 39 for Barker 13 signal

designed to have its sidelobe peaks at different locations than those of the first filter. Section 6 discusses how such a pair of filters can be designed. Fig. 2 (top) displays the response of two contrasting mismatched filters of length 39, designed for Barker 13 signal. The bottom subplot demonstrates how the response difference always exceeds 9 dB. The top subplot displays also the response of the minimum integrated sidelobes (ISL) filter of length 39. It shows that the contrasting filters are only slightly worse. The parameters a and b are design parameters in the generation

of the contrasting filters. Their exact meaning will be explained in Section 6.

Our sidelobe-blanking approach bears some resemblance to the ‘apodization’ method [3]. This method is used to counteract the mainlobe spreading seen when using weighting schemes to lower sidelobes. The received signal is processed by two channels, in one of which weighting (e.g. Hann) is applied and in the other no weighting is applied. The results are compared, and detection decisions

made on the basis of this comparison. In this way, one can obtain the sidelobe advantage of weighting while also getting the benefits of the narrower unweighted response.

The simple blanking logic is graphically described in Fig. 3. If the auxiliary (guard) output v is nearly equal to the main output u , namely $vF < u < v/F$, $F < 1$, then u is not blanked and passes on to the detection threshold test. We will refer to F as ‘agreement factor’. The pass rule PASS is therefore

$$\text{PASS} = \begin{cases} 1, & vF < u < v/F \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

The detection decision D can be summarised as follows

$$D = \begin{cases} 1, & \max(u_{\text{th}}, vF) < u < v/F \\ 0, & \text{elsewhere} \end{cases} \quad (2)$$

where u_{th} is the detection threshold. Choosing $F = 0.63$ for example, implies that the difference between the two outputs must be within ± 2 dB (note that F is a power ratio).

3 Blanking a target

The sidelobe-blanking algorithm can also blank targets. Blanking a target return at a given delay depends on the two sidelobe levels at that delay, u_{SL} , v_{SL} , on the target level s and on the agreement factor F . For a given delay we will define the following two power ratios

$$R = \frac{\overline{u_{\text{SL}}^2}}{\overline{v_{\text{SL}}^2}} \quad (3)$$

which is the ratio between the average sidelobe power level at the outputs of the two filters (the ‘contrast’), at the given delay, and

$$K = \frac{\overline{s^2}}{\overline{u_{\text{SL}}^2}} \quad (4)$$

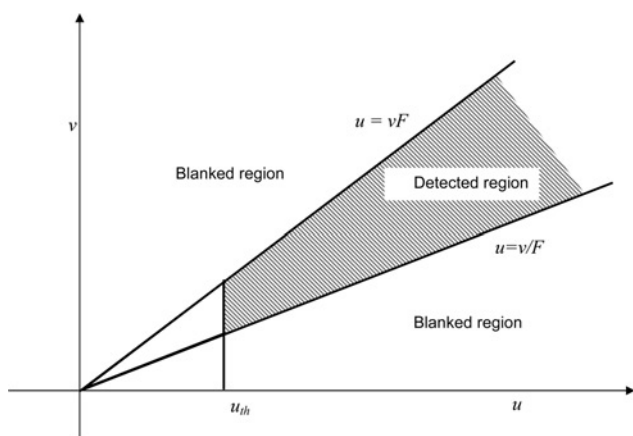


Figure 3 Blanking logic

which is the average signal-to-sidelobe power ratio (at a given delay) between the signal output of either filter (both are inherently equal) and the sidelobe level of output u . Using these ratios the two average outputs at a given delay are given by

$$\overline{u} = \overline{(s + u_{\text{SL}})^2} = \overline{s^2} + \overline{u_{\text{SL}}^2} \quad (5)$$

$$\overline{v} = \overline{(s + v_{\text{SL}})^2} = \overline{s^2} + \overline{v_{\text{SL}}^2} \quad (6)$$

where we have assumed that the signal’s complex value and the sidelobe’s complex value, at a given delay, which are the results of two different targets, are independent and their phases are uniformly distributed between 0 and 2π . Applying (3) and (4) to the pass rule (1), yields

$$\overline{\text{PASS}} = \begin{cases} 1, & F < \frac{K+1}{K+1/R} < \frac{1}{F} \\ 0, & \text{elsewhere} \end{cases} \quad (7)$$

The resulted pass rule as function of R and K , for the case $1 < R$ and $10 \log_{10}(1/F) = 2$ dB, is demonstrated in Fig. 4. The ‘pass’ region is to the right of the 2 dB contour, which represents the required agreement between the two filter outputs to be $F_{\text{dB}} = 10 \log_{10}(1/F) = 2$ dB. Thus, when R is 5 dB and, a target whose relative strength is $K = 4$ dB above the sidelobe level (filled circle) will pass through the blanking algorithm, but a target whose level is 4 dB below the sidelobe (empty circle) will be blanked.

A more detailed blanking diagram is given in Fig. 5, that uses three values of F_{dB} (4, 2 and 0.5 dB). It shows that, for a given R , as F_{dB} gets closer to 0 dB, stronger targets are blanked. Note also that as K decreases (practically no target, just sidelobes) setting F smaller than R will cause blanking, which is the main purpose of this algorithm.

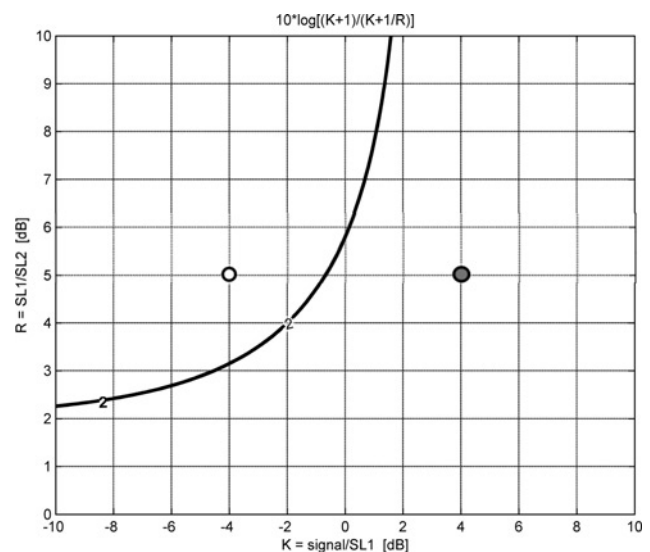


Figure 4 Blanking diagram ($1 < R$)

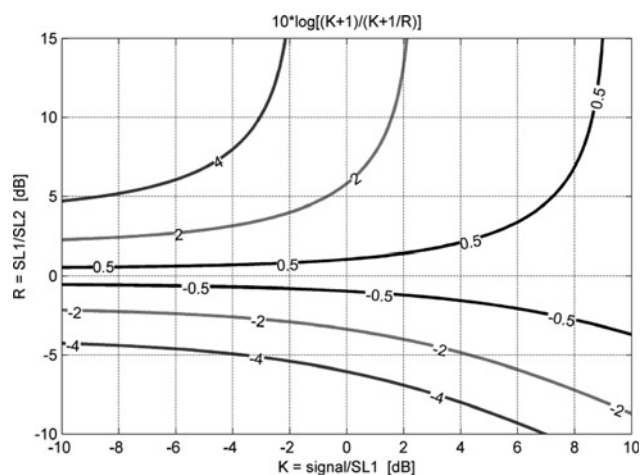


Figure 5 Blanking diagram

4 Over-sampling

A sequence is not yet a signal. A complex element of the sequence represents the magnitude and phase during the duration t_b of an element of the signal. The interval between samples t_s in the radar receiver is likely to be shorter. We will assume an integer number of samples within a signal element, namely $t_b = mt_s$. In order to simulate the over-sampled processor, each element of the transmitted sequence and each element of the mismatched filter will be repeated m times. Having m samples per signal and per filter element will also reveal the asymmetry of the response of a mismatched filter (to a binary signal) around the mainlobe, even when the logarithm of the magnitude is displayed. Fig. 6 compares the response of a

minimum ISL filter of length 39 to a Barker 13 signal, with and without over-sampling. The over-sampling (bottom subplot) reveals many polarity reversals on the left-hand side of the response, which are missed when the response is plotted in dB using a single sample per bit (top subplot).

Over-sampling has an important role in our proposed sidelobe-blanking scheme. Note from the lower subplot of Fig. 6 that with over-sampling ratio m the mainlobe width is approximately $2m$. Thus, even a point target (the narrowest possible) will result in a response $2m$ samples wide. Narrower peaks, especially single-sample peaks, are likely to be artefacts of the blanking scheme, hence ought to be blanked.

5 Preliminary simulation

The blanking concept is demonstrated with a Barker 13 signal. The noise-free scene of reflections is plotted in the top subplot of Fig. 7. Each reflector may extend over one or more delay samples. Each reflector is defined by both intensity and phase. The intensities are preset, but the phase (of each delay sample of the reflector) is randomly distributed with a uniform distribution over 2π . With the over-sampling ratio m , we are able to simulate what happens to targets whose delay width is narrower than the signal's delay resolution. Fig. 7 was obtained with oversampling ratio of $m = 4$. As explained in the previous section, over-sampling is simulated by repeating each signal and each filter element m times.

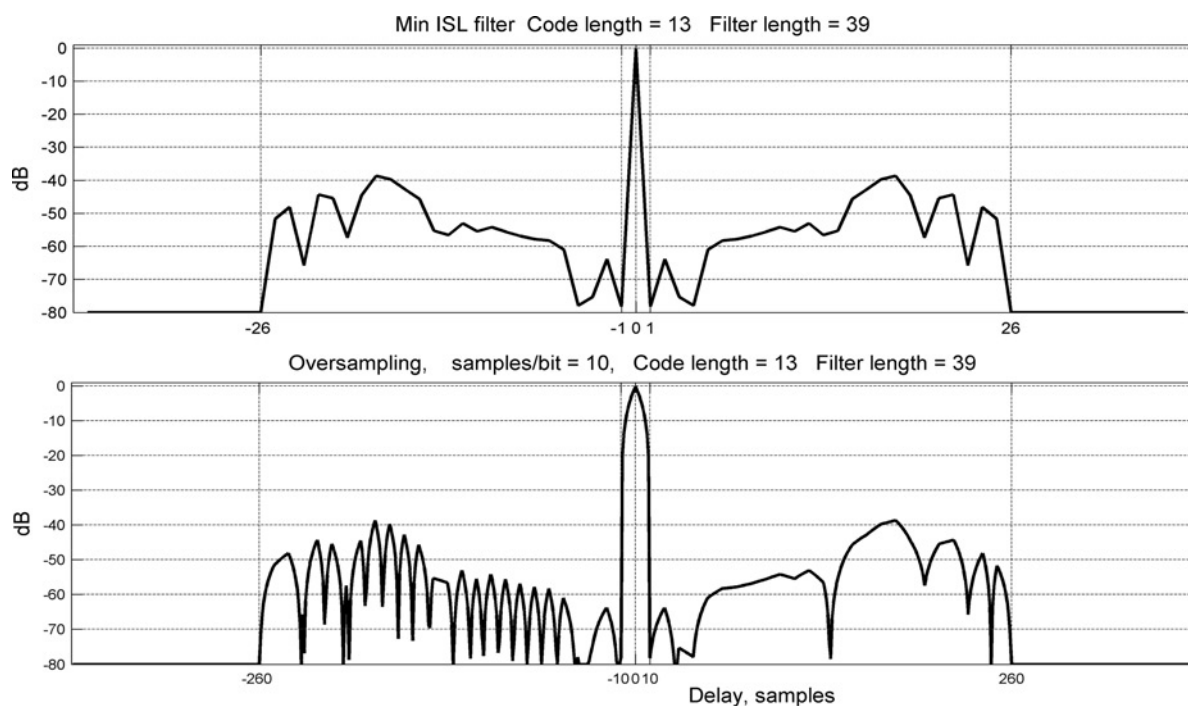


Figure 6 Minimum ISL response without (top) and with over-sampling (bottom)

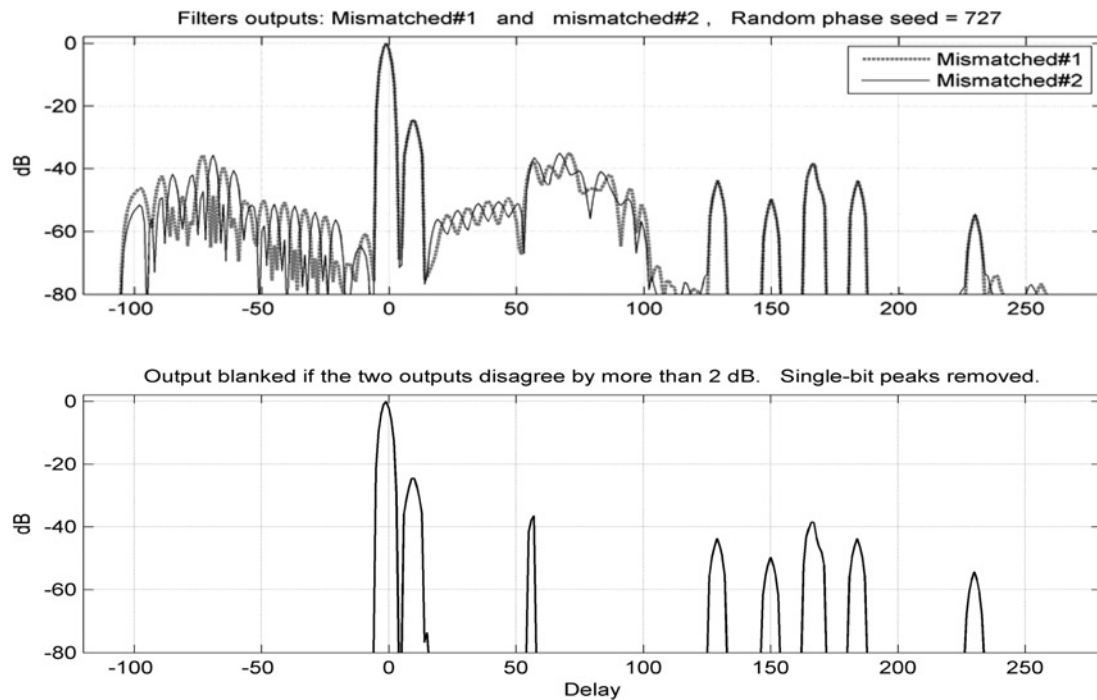


Figure 8 The two filter outputs (top) and the resulted blanked output (bottom)

for example

$$\begin{aligned}
 \text{diag}(W1) &= [\dots bbbbaaaaaabb0 \\
 &bbbaaaabbbb\dots] \\
 \text{diag}(W2) &= [\dots aaaaaabbbbbaa0 \\
 &aabbbbbaaaaa\dots]
 \end{aligned}
 \tag{9}$$

The result responses are shown in Fig. 10. Repeating weight elements when designing the mismatched filter should not be confused with repeating the filter elements after it was designed. Comparing the lower subplots of Figs. 2 and 10 demonstrates that repeating weights raise the response differences at the near sidelobes, but lowers them (below 5 dB) at the far sidelobes, which can cause insufficient

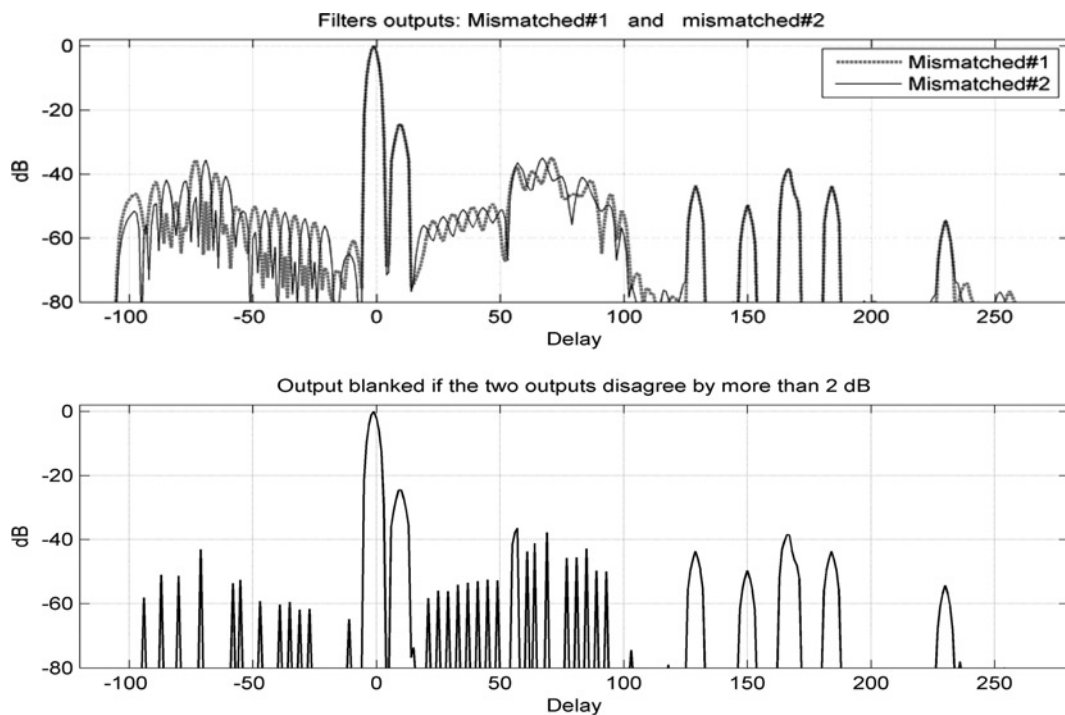


Figure 9 The two filter outputs (top) and the resulted blanked output (bottom) before removing single-sample peaks

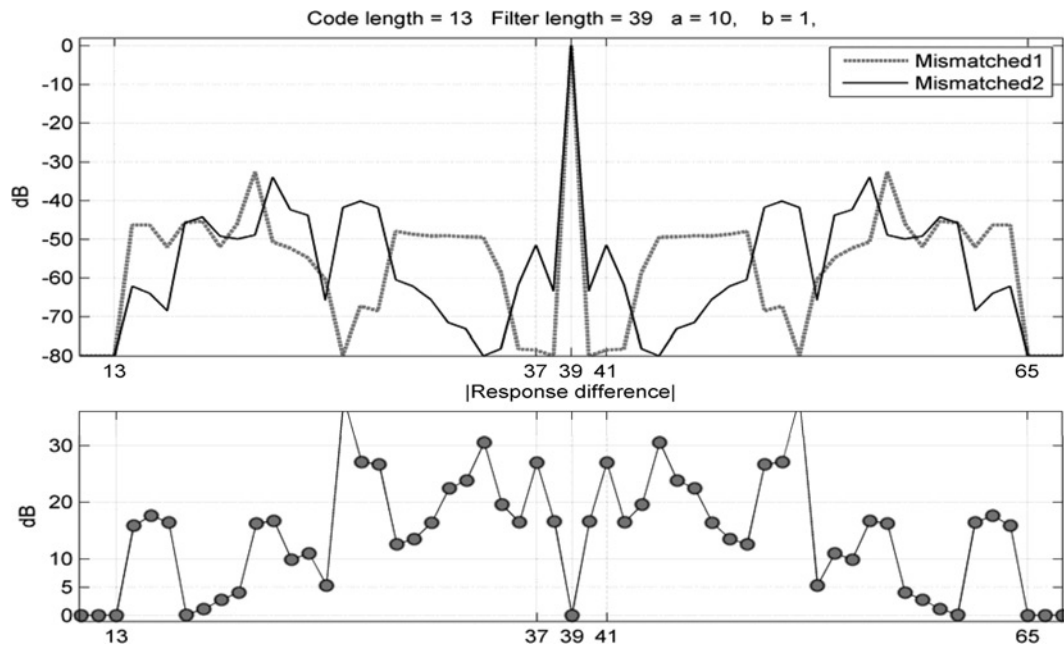


Figure 10 Responses of two contrasting mismatched filters of length 39 for Barker 13 signal, designed using repeated weight elements according to (9)

blanking of the far sidelobes. Owing to symmetry the roles of a and b can be interchanged.

It would be desired that the response differences (in dB) between the sidelobes at each given delay of the two filters should be greater than F_{dB} . Sidelobes whose differences are greater than F_{dB} are blanked. Designing two filters that have sidelobes response differences that are greater than F_{dB} for all delays is not always possible. Usually, designs have a few delays with sidelobe response differences that are smaller than F_{dB} . When a sidelobe, with response difference smaller than F_{dB} , is located between two sidelobes that have a response difference greater than F_{dB} , it might result in a single-sample output peak.

Single sample peaks are likely to be removed by the blanking algorithm. However, if there are consecutive response differences smaller than F_{dB} they may result in two or more consecutive false output peaks, which will not be blanked, thus resulting in a false target. Therefore the design should avoid consecutive responses difference smaller than F_{dB} .

Crossings between the two filter responses might also create false targets. In general, it would therefore be desired that the design yields as little possible response difference smaller than F_{dB} and as little possible crossings. Finally, at those delays where one of those two undesired cases do occur, it would be preferred that the response sidelobe levels would be as low as possible, so that the resulted false target level will be below the detection threshold level. Note in Fig. 10 that, while there are fewer crossings, there are some consecutive far-sidelobes whose differences are

small, and at the same time their level is high; an undesired combination. We found that this is typical of designs with repeated weight elements.

On the other hand, the response differences much greater than F_{dB} may imply that one of the two sidelobes (at that delay) is high and may blank a collocated small target. So there is a tradeoff between false alarm and misdetection that must be handled. The way to set the line between the two is to aim for response differences higher than F_{dB} by a factor. The factor should be optimised according to the system requirements.

7 Blanking with two channels

Comparing Fig. 10 with Fig. 2 prompts the following question: if the design of a contrasting pair of filters based on the alternating weight elements [Figs. 2 and (8)] behaves better than a design based on repeated weight elements [Figs. 10 and (9)], why not drop the second approach. The answer is that we can obtain improved blanking by using two parallel pairs of filters, namely two parallel blanking channels. The overall output at each delay will be based on an AND decision rule between the two outputs. Only outputs that were not blanked in both channels will pass on to the threshold stage. Obviously we would like the two channels to have good but different responses. The two different designs of the weight elements (alternating or repeating) yield the desired different responses.

The advantage of using two parallel blanking channels will be demonstrated by simulating a denser target scene. The

signal is Barker 13 and the filters are of length 39. The over-sampling ratio is $m = 3$ producing 39 signal samples and 117 filter samples. The filter responses are plotted in Fig. 11. The weights used in the filters design were as follows

Filter pair #1 : $a = 1, b = 10$

$\text{diag}(W1) = [\dots\dots b a b a 0 b a b a \dots\dots]$

$\text{diag}(W2) = [\dots\dots a b a b 0 a b a b \dots\dots]$

Filter pair #2 : $a = 1, b = 30$

$\text{diag}(W1) = [\dots\dots b b a a b b a a 0 b b a a b b a a \dots\dots]$

$\text{diag}(W2) = [\dots\dots a a b b a a b b v 0 a a b b a a b b \dots\dots]$

Blanking that requires agreement between two pairs, allows some relaxation of the agreement factor within each pair. Hence $F_{\text{dB}} = 3\text{dB}$ was selected. Fig. 12 displays the noise-free detection performances. The top subplot presents the intensities and relative phases of the 13 target reflections. The second subplot displays the output of one of the mismatched filters. The dots mark delays of true targets. The third and fourth subplot display the blanked outputs of the two channels. Note that each output contains few false targets. The lowest subplot displays the final output, that is free of false targets. The example demonstrates the advantage of using two blanking channels in parallel. It is possible to use more parallel channels with diminishing further improvement.

8 Noise and integration

The simulations in the previous section were repeated with additive white Gaussian noise. Because the same noise enters all the blanking filters there is no reason to expect noise blanking. However, in delays where the noise is weaker than the sidelobes (e.g. around the strong targets on the left-hand side of Fig. 13) it may be blanked together with the sidelobes. While in delays where the noise is stronger or of the same level as the sidelobes (e.g. around the weak targets on the right hand side of Fig. 13) the noise does show up at the output.

So far our examples assumed single pulse detection. However, the blanking concept can be combined with pulse integration. Both coherent and non-coherent integration can be used. Since the blanking processor performs a nonlinear operation, it should usually follow pulse integration. Coherent integration involves Doppler processing, hence the blanking processor should be performed for each Doppler of interest. Non-coherent integration is likely to have a single output in which the blanking processor should operate. As far as the blanking stage is concerned, the signal after integration is basically a signal with lower noise.

9 Example with a longer code sequence

In this section, we will simulate two-channel blanking using an MPSL 51 signal [5] (See also Table 6.3 in [6]).

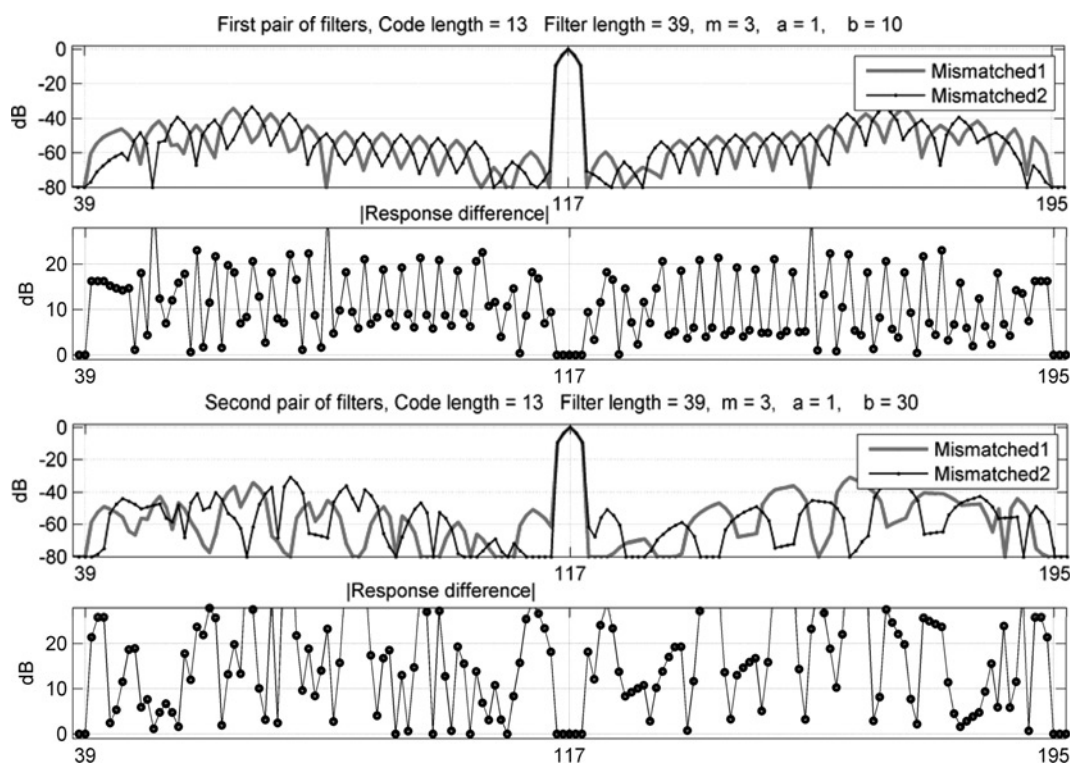


Figure 11 Responses of two pairs of mismatched filters

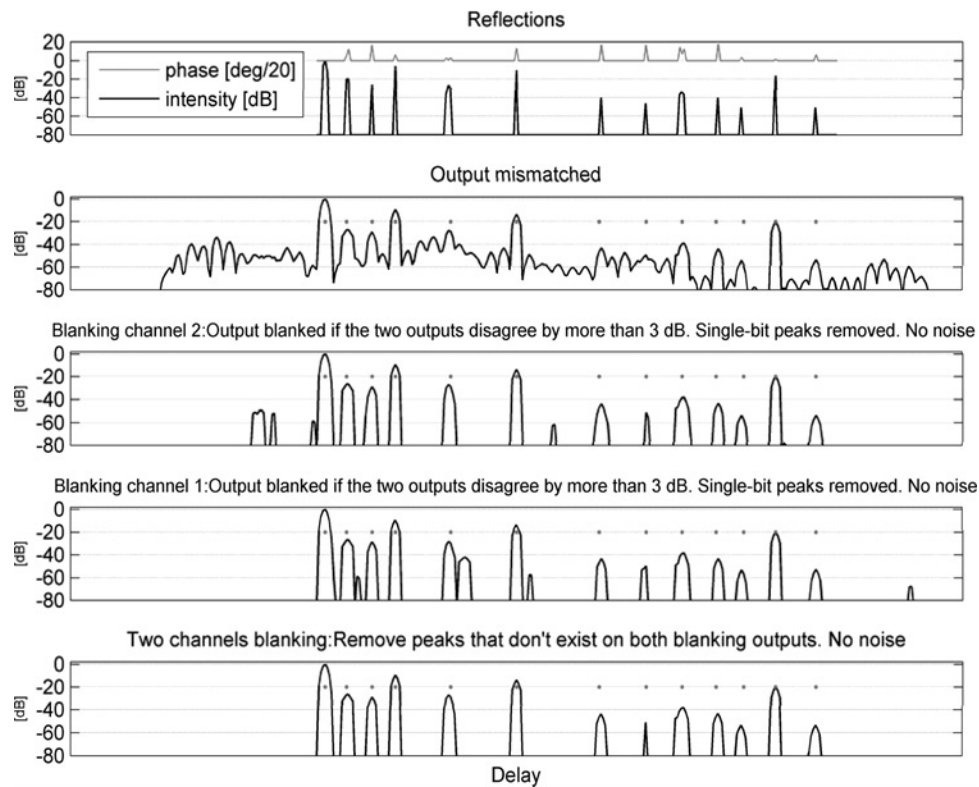


Figure 12 Detection performances using two pairs of mismatched filters

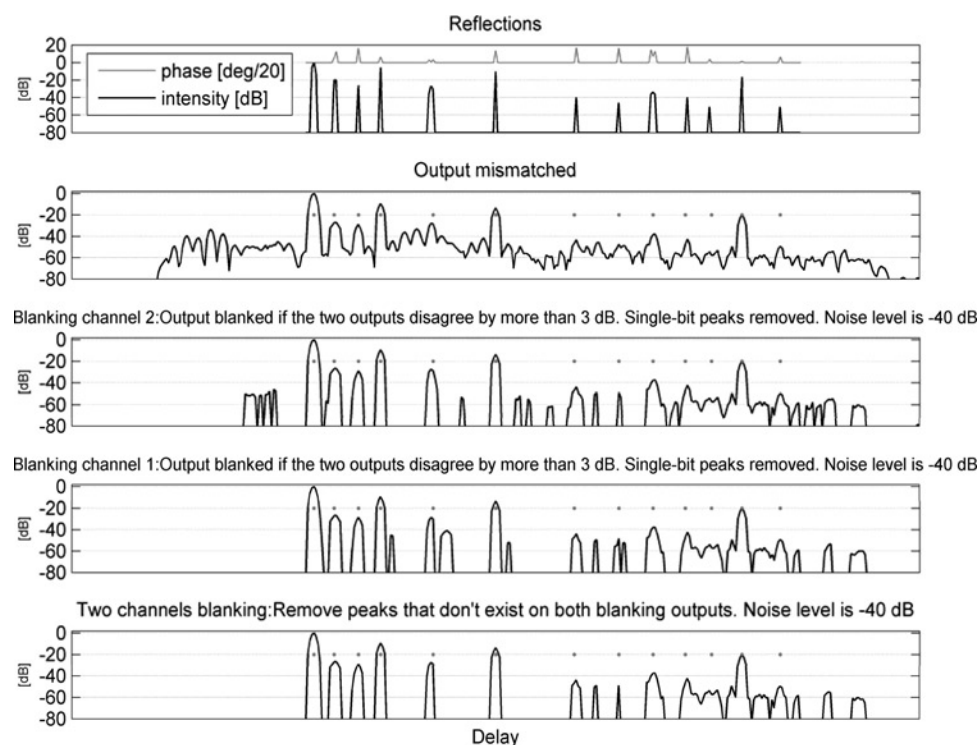


Figure 13 Detection in the presence of noise (using two pairs of mismatched filters)

This is the longest binary code with peak autocorrelation sidelobe of 3, when the mainlobe height equals the code length ($= 51$). The mismatched filters will be designed to be nine times longer than the signal. This

requires two filters for each blanking channel, each filter 459 elements long. The over-sampling ratio will be $m = 3$, resulting in 153 signal samples and 1377 filter samples. The filter-pair responses and response differences

appear in Fig. 14. The sidelobe weightings are listed below.

First pair: $a = 1, b = 8$

```
diag(W1) = [... bbbbbbbbbbbbbbbbbbbba
aaaaaaaaaaaaaaaaabbbbbbb0aaaaaaaaab
bbbbbbbbbbbbbbbbbaaaaaaaaaaaaaaaaa
... ]
diag(W2) = [... aaaaaaaaaaaaaaaaaabbbbbbb
bbbbbbbbbbbbbaaaaaaaaa0bbbbbbbbbaaaaaaa
aaaaaaaaabbbbbbbbbbbbbbbbbbb ... ]
```

Second pair: $a = 1, b = 8$

```
diag(W1) = [.bbbbbbbbbbbbbbbbbb
aaaaaaaaaaaaaaaaabbbbbbbbbbb0
aaaaaaaaaaaaaaaaabbbbbbbbbbbba
aaaaaaaaaaaaaaaaa... ]
diag(W2) = [.aaaaaaaaaaaaaaaaabbbbbbb
bbbbbbbbbbbbbaaaaaaaaa0bbbbbbbb
bbbbbbbbbaaaaaaaaaaaaaaaaaabbbbbbb
bbbbbbbbbb ... ]
```

While cross-correlation does not have to be symmetrical, the mismatched minimum ISL filter responses, seen in Fig. 14,

are considerably non-symmetrical. Their slanted response was observed with other MPSL signals, but not in their immediate neighbour sequences (MPSL 50 and MPSL 52). On the other hand, the mismatched minimum ISL response of MPSL 51 exhibits much lower SNR loss than its neighbours.

The benefits of using a long sequence, long filters and two-channel blanking will be demonstrated by defining a 16-targets scenario (Fig. 15). The targets are relatively close to each other and every target is within the sidelobes span of other targets. Furthermore, the power differences between targets may exceed 60 dB. The second subplot of Fig. 15 shows that despite the low sidelobe level of a single mismatched filter (< -50 dB), many of the 16 targets are undistinguishable from sidelobes. The output of one of the blanking channels (fourth subplot) does contain two false targets. This would have been the final output if it was the only blanking channel. The AND decision rule, applied to the two blanking channels, resulted in the final output (bottom subplot), which contains all the targets and only them.

10 Doppler tolerance

When the compressed pulse is part of a coherent pulse train used in pulse-Doppler radar, there is a concern about the Doppler tolerance of the pulse compression. In order to

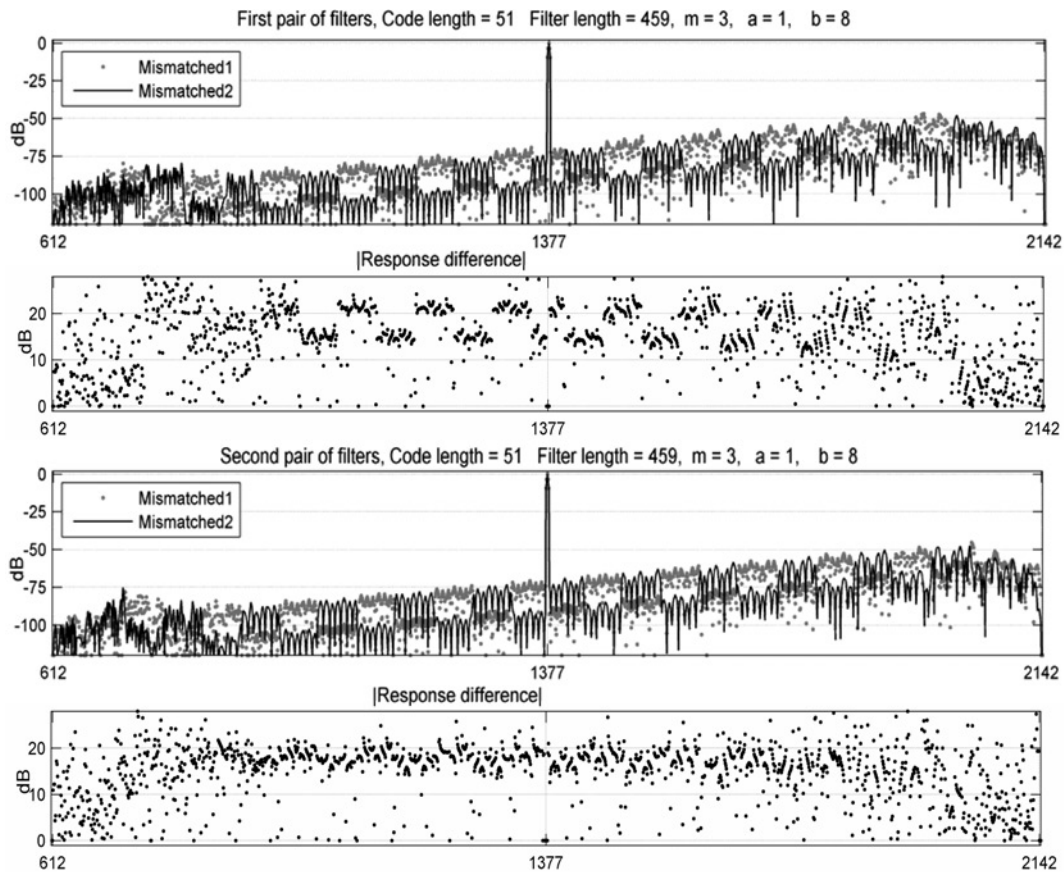


Figure 14 Responses of mismatched filters for MPSL 51 sequence

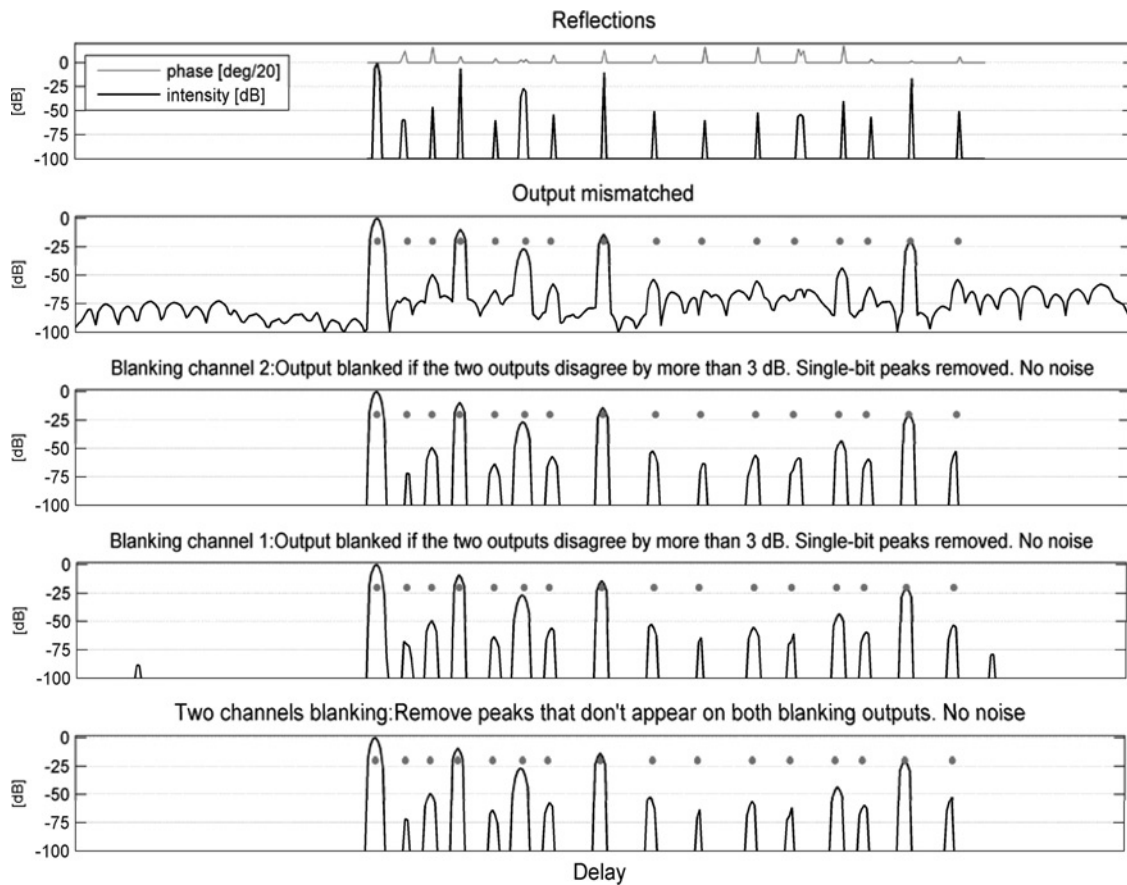


Figure 15 Detection performances using two pairs of mismatched filters for MPSL 51 sequence

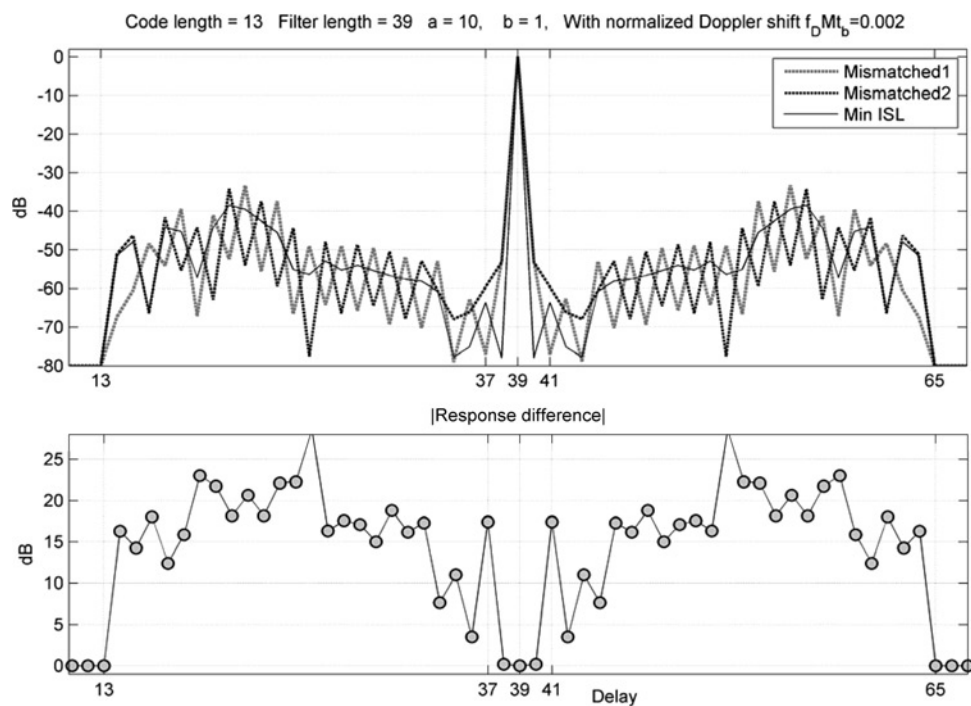


Figure 16 Repeat of Fig. 2 with Doppler shift

match the processor to a non-zero Doppler shift, the processor compensates the inter-pulse phase ramp caused by that Doppler shift. However, intra-pulse compensation is rarely performed. This may cause a drop in the mainlobe of the delay response, as well as a rise in the sidelobes. Because the main purpose of mismatched processing is to lower the sidelobes, their increase because of Doppler is of special concern. The same concern now applies to how well sidelobe blanking is maintained. In order to demonstrate the effect of Doppler, the filters used in Fig. 2 processed a signal with a Doppler shift. The result is shown in Fig. 16.

The normalised Doppler used in Fig. 16 was $f_D M t_b = 0.002$, where f_D is the actual Doppler shift and $M t_b$ is the length of the uncompressed pulse. If the uncompressed pulse duration is $0.67 \mu\text{s}$, then a 0.002 value of normalised Doppler corresponds to a 3000 Hz actual Doppler. How that relates to range-rate will depend on the radar wavelength. Comparing Figs. 2 and 16 show that the Doppler shift raised the near-sidelobes and modified their response difference. Fortunately, the near sidelobes of minimum ISL filters are inherently very low, compared with the far sidelobes. Hence, even if their difference became too small to activate the blanking mechanism, their low level makes them of little concern.

11 Conclusions

A new concept for blanking range sidelobes was proposed. It is based on comparing parallel outputs of different mismatched filters, designed to have their sidelobe peaks at different delays. An agreement of the outputs at a given delay indicates a true target return. A disagreement hints that the output at that delay is due to a sidelobe. A brief description of the concept was given, accompanied by encouraging preliminary simulations.

We have presented a new concept and experiments are invited to test its feasibility. There is also place for further work on detection performances in the presence of noise, as well as on combining sidelobe blanking with CFAR detection.

12 References

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13 Appendix

13.1 Filter design

Let $c[n]$ be the code sequence of length N , $b[n]$ the reference sequence in the receiver (of length P) and $y[n]$ the output of the correlator. If $b[n]$ is the conjugate of the code $c[n]$ then it is a matched filter. When $N < P$ we will zero-pad the code to reach the same length as the filter. The zero-padded code will be labelled x , hence

$$\mathbf{x}^T = [0 \ 0 \ c_0 \ c_1 \ \dots \ c_{N-1} \ 0 \ 0] \quad (10)$$

$$\mathbf{b}^T = [b_0 \ b_1 \ \dots \ b_{P-1}] \quad (11)$$

where $()^T$ denotes transpose operation. The output of the correlator is

$$y_k = \sum_{n=0}^{P-1} x_n b_{n-k}^* \quad k = -(P-1), \dots, (P-1) \quad (12)$$

This cross-correlation function can be represented in matrix form as

$$\mathbf{y} = \mathbf{b}^T \boldsymbol{\Psi} \quad (13)$$

where $()^T$ denotes conjugate transpose operation and $\boldsymbol{\Psi}$ is a $P \times (2P-1)$ Hankel matrix of x .

$$\boldsymbol{\Psi} = \begin{bmatrix} 0 & 0 & \dots & x_{P-2} & x_{P-1} \\ 0 & 0 & \dots & x_{P-1} & 0 \\ 0 & x_0 & \dots & 0 & 0 \\ x_0 & x_1 & \dots & 0 & 0 \end{bmatrix} \quad (14)$$

The important features of the cross-correlation output are its peak and sidelobes. The peak is related to the SNR loss of the system. The larger the peak is, the easier it is to detect the target. The sidelobes are an undesirable by-product of pulse compression. The main goal of mismatched filters design is to reduce the sidelobes while keeping the peak high. An important quality measure is the ISL energy ratio

$$\text{ISLR} = \frac{1}{y_0^2} \sum_{k \neq 0} y_k^2 \quad (15)$$

This measure will be used in designing the minimum ISLR mismatched filter.

13.2 Minimum ISL mismatched filter

The total energy E in the range sidelobes can be expressed as

$$E = \mathbf{y} \mathbf{W} \mathbf{y}^{\Gamma} \quad (16)$$

where \mathbf{W} is a $(2P - 1) \times (2P - 1)$ diagonal matrix, which will be referred to as the weight matrix. It is an Identity matrix except for the middle element, which is zero. Substituting (13) into (16) gives

$$E = (\mathbf{b}^{\Gamma} \boldsymbol{\Psi}) \mathbf{W} (\mathbf{b}^{\Gamma} \boldsymbol{\Psi})^{\Gamma} = \mathbf{b}^{\Gamma} (\boldsymbol{\Psi} \mathbf{W} \boldsymbol{\Psi}^{\Gamma}) \mathbf{b} = \mathbf{b}^{\Gamma} \mathbf{B} \mathbf{b} \quad (17)$$

The unnormalised filter \mathbf{b}_0 , which solves the minimisation

problem, is given by

$$\mathbf{b}_0 = \mathbf{B}^{-1} \mathbf{x} \quad (18)$$

The filter is then normalised to have the same energy as a matched filter, yielding

$$\mathbf{b} = \mathbf{b}_0 \sqrt{\frac{\mathbf{x}^{\Gamma} \mathbf{x}}{\mathbf{b}_0^{\Gamma} \mathbf{b}_0}} \quad (19)$$

The diagonal of the weight matrix \mathbf{W} can be used to assign different weights to different sidelobes. This feature was used in the paper to create responses with contrasting sidelobe locations.