Some Results From Utilizing Doppler Derivatives

Abstract

Explicit expressions of range, velocity, and the angle between them, as functions of a radar Doppler shift and its first two derivatives are given.

I. Analysis

The Doppler shift as a function of time of a static radar return from a moving target (or vice versa) is considered. The target is assumed to be moving in a straight line, at a constant velocity v. Usually, the first two terms of the Taylor series are given [1]. The first three terms of Taylor's expansion of the Doppler shift, $f(t)$, about $t = 0$, are

$$f(t) \cong (2v/\lambda)\cos\theta_0 - (2v^2 t/\lambda R_0)\sin^2\theta_0 - (3v^3 t^2/\lambda R_0^2)\cos\theta_0 \sin^2\theta_0 \quad (1)$$

where R_0 is the range, θ_0 is the angle between the range and the velocity vectors, both at $t = 0$, λ is the radar wavelength.

From (1) we get

$$f \triangleq f(t)\big|_{t=0} = (2v/\lambda)\cos\theta_0 \quad (2)$$

$$\dot{f} \triangleq d[f(t)]/dt\big|_{t=0} \cong -(2v^2/\lambda R_0)\sin^2\theta_0 \quad (3)$$

$$\ddot{f} \triangleq d^2[f(t)]/dt^2\big|_{t=0} \cong -(6v^3/\lambda R_0^2)\cos\theta_0\sin^2\theta_0\,. \quad (4)$$

Equations (2)-(4) are in agreement with Barton [2] when the proper substitutions are made.

In (2), (3), and (4), f, $\dot{f}$, and $\ddot{f}$ are given as functions of R_0, v, and θ_0. It can easily be shown that the latter can be explicitly obtained as functions of the former, namely

$$R_0 \cong (3/2)\lambda\,(f\dot{f}/\ddot{f}) \quad (5)$$

$$v \cong (\lambda/2)\,(f^2 - 3\dot{f}^2 f/\ddot{f})^{1/2} \quad (6)$$

$$\theta_0 \cong \cos^{-1}(1 - 3\dot{f}^2/f\ddot{f})^{-1/2}\,. \quad (7)$$

While obvious, it may be helpful to add here that the derivatives could be obtained from three or more equally-space Doppler shift measurements $f(0), f(\tau), f(2\tau), \ldots$.

Three measurements yield

$$\begin{bmatrix} f \\ \dot{f}\tau \\ \ddot{f}\tau^2/2 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ -3/2 & 2 & -1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{bmatrix} f(0) \\ f(\tau) \\ f(2\tau) \end{bmatrix}. \quad (8)$$

Four measurements yield

$$\begin{bmatrix} f \\ \dot{f}\tau \\ \ddot{f}\tau^2/2 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 1 & -5/2 & 2 & -1/2 \end{bmatrix} \cdot \begin{bmatrix} f(0) \\ f(\tau) \\ f(2\tau) \\ f(3\tau) \end{bmatrix}. \quad (9)$$

The estimation filters given in (8) and (9) are based on Taylor expansions. Another approach is based on a second-order polynomial, constructed from linear combination of discrete Legendre polynomials, which best fits the data vector in the sense of least squares [3]. The Legendre polynomials for three measurements yield a filter identical to (8), The filter for four measurements is given in (10).

$$\begin{bmatrix} f \\ \dot{f}\tau \\ \ddot{f}\tau^2/2 \end{bmatrix} = (1/20) \begin{bmatrix} 19 & 3 & -3 & 1 \\ -21 & 13 & 17 & -9 \\ 5 & -5 & -5 & 5 \end{bmatrix} \cdot \begin{bmatrix} f(0) \\ f(\tau) \\ f(2\tau) \\ f(3\tau) \end{bmatrix}. \quad (10)$$

While (10) is more efficient in the sense of least squares, (9) is considerably better, in our application, with regard to systematic error. Equation (9) will therefore be utilized in the remaining of this work.

II. Errors and Applications

The dependence of the random error in the estimation of R_0, v, and θ_0, on the random error in the measurements of the Doppler frequencies, is inversely related to the in-

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terval τ (see Appendixes A and B). Furthermore, the estimation error of the Doppler frequency is itself inversely dependent on the frequency measuring period [4], which has to be smaller than τ. Thus the random error strongly decreases as τ increases. There is, however, a limit on τ, dictated by the systematic error.

The systematic error is due partly to the approximations in the truncated expansions used in (8), (9), or (10). Additional systematic error may be the result of a wrong model, e.g., if the true motion is not along a straight line or not at a constant velocity. Both systematic errors increase with τ.

In considering applications of the explicit algorithm for obtaining R_0, v, and θ_0, as expressed in (5)-(7), this algorithm has to be compared to iterative algorithms which are normally used. Iterative algorithms can utilize more complicated motion models, and a longer section of the target pass, with many more measurements. The "redundant" measurements allow lower signal-to-noise ratio (SNR) during each frequency measurement. In such low SNR applications our explicit algorithm can probably serve only as a fast mean to obtain an initial guess for a more elaborate iterative algorithm.

Our explicit algorithm can stand alone in short range (high SNR) applications, e.g., muzzle velocity radar (angle independent). In such applications the high SNR can reduce random error despite the short frequency measurement period, which is necessary in order to avoid systematic error. In Appendix B we demonstrate the systematic error in a numerical example.

The Appendixes show that the explicit algorithm is both simple and, in high SNR applications, sufficiently accurate. It can thus yield accurate velocity without a priori knowledge of the angle.

Appendix A

Random Error Analysis

We define the measurement vector

$$\mathbf{F} = [f(0), f(\tau), f(2\tau), f(3\tau)]^{\mathrm{T}} \tag{A1}$$

and the parameter vector

$$\mathbf{S} = (R_0, v, \theta_0)^{\mathrm{T}} \ . \tag{A2}$$

If our estimate $\hat{\mathbf{S}}$ is in the neighborhood of the true value $\mathbf{S}$, we can make the approximation

$$\hat{\mathbf{S}} = \mathbf{S} + D^{\mathrm{T}}(\hat{\mathbf{F}} - \mathbf{F}) \tag{A3}$$

where D is a 4×3 matrix of partial derivatives

$$D = \begin{bmatrix} \partial R_0/\partial f(0) & \partial v/\partial f(0) & \partial\theta_0/\partial f(0) \\ \partial R_0/\partial f(\tau) & \partial v/\partial f(\tau) & \partial\theta_0/\partial f(\tau) \\ \partial R_0/\partial f(2\tau) & \partial v/\partial f(2\tau) & \partial\theta_0/\partial f(2\tau) \\ \partial R_0/\partial f(3\tau) & \partial v/\partial f(3\tau) & \partial\theta_0/\partial f(3\tau) \end{bmatrix} \tag{A4}$$

Using (A3) we get the error convariance matrix of $\hat{\mathbf{S}}$ from the error covariance matrix of $\hat{\mathbf{F}}$

$$\mathrm{cov}(\hat{\mathbf{S}}) = D^{\mathrm{T}} \, \mathrm{cov}(\hat{\mathbf{F}}) \, D \ . \tag{A5}$$

It is reasonable to asume that

$$\mathrm{cov}(\hat{\mathbf{F}}) = \sigma_f^2 \, I \tag{A6}$$

where I is the 4×4 identity matrix, and σ_f is the root-mean square (rms) error in the Doppler shift measurements.

Using range as an example, we define

$$\mathbf{D}_R = [\partial R_0/\partial f(0), \partial R_0/\partial f(\tau), \partial R_0/\partial f(2\tau), \partial R_0/\partial f(3\tau)]^{\mathrm{T}} \tag{A7}$$

Thus (A5), (A6), and (A7) yield

$$\sigma_R = \sigma_f (\mathbf{D}_R^{\mathrm{T}} \, \mathbf{D}_R)^{1/2} \tag{A8}$$

where σ_R is the rms error in range estimation. Similar equations can be written for σ_v and σ_θ.

From (5) and (9) it can be shown that

$$\mathbf{D}_R = (\lambda f/4\tau\ddot{f} \begin{bmatrix} -11 & -4 & 6 \\ 18 & 10 & 0 \\ -9 & -8 & 0 \\ 2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3\dot{f}/\tau\ddot{f} \\ \dot{f}\tau/f \end{bmatrix} \ . \tag{A9}$$

Similarly, from (6) and (9) it can be shown that

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$$
\mathbf{D}_v = (\lambda^2 f\dot{f}/8v\tau\ddot{f}) \begin{bmatrix} 11 & 2 & [(2f\ddot{f}/\dot{f}^2) - 3] \\ -18 & -5 & 0 \\ 9 & 4 & 0 \\ -2 & -1 & 0 \end{bmatrix}
\cdot \begin{bmatrix} 1 \\ 3\dot{f}/\tau\ddot{f} \\ \dot{f}\tau/f \end{bmatrix} . \tag{A10}
$$

If σ_f, the rms frequency error, is known, then (A8) and (A9) provide the rms of the random range error. Similarly, using (A10), we can get the rms of the random velocity error.

Appendix B

Systematic Errors and a Numerical Example

The systematic error does not yield itself to an analytic analysis as the random error does. Therefore, we use the following numerical example, which may correspond to a muzzle velocity radar

$$\mathbf{S} = (10\text{ m}, 200\text{ m/s}, 150^\circ)^{\mathrm{T}}, \quad \lambda = 0.02\text{ m}.$$

The true (measured) Doppler shifts were calculated using equations of motion which included linear acceleration, a.

$$f(t) = [2v(t)/\lambda] \cos[\theta(t)] \tag{B1}$$

$$v(t) = v_0 + at \tag{B2}$$

$$\sin[180^\circ - \theta(t)] = [R_0/R(t)] \sin\theta_0 \tag{B3}$$

$$R(t) = [R_0^2 + (v_0 t + at^2/2)^2 - 2(v_0 t + at^2/2)R_0 \cos\theta_0]^{1/2} . \tag{B4}$$

The velocity estimation was calculated using (9) and (6), for various τ, with and without deceleration of 20 m/s^2. The results appear in Table I.

The velocity error in the column corresponding to $a = 0$ is due to the truncated Taylor series in (9). The error increases rather rapidly with τ. The additional error due to a deceleration of 20 m/s^2 is relatively small and in the opposite direction. It should be pointed out that the systematic error decreases at the same ratio in which R_0 is increased (at $a = 0$ and all other parameters unchanged). On the other hand, the additional error due to the deceleration increases at the same ratio in which R_0 is increased.

TABLE I

Calculated Velocity and σ_v/σ_f for Various τ and a (True Velocity is 200 m/s)

τ (s)	v in (m/s), $a = 0$	v in (m/s), $a = -20$ (m/s^2)	σ_v/σ_f [(m/s)/Hz]
0.005	201.63	200.58	0.95
0.01	204.52	203.36	0.15
0.1	226.30	221.88	0.014

The rms random velocity error was also calculated for this numerical example using (A10). It appears in the last column of Table I. At $\tau = 0.01$ s, for example, if the SNR is high enough to limit σ_f to 10 Hz (at a Doppler frequency of 17 kHz and a measuring period shorter than 10 ms), then the rms random velocity error will be 1.5 m/s, which is smaller than the corresponding systematic error. The preceding example was repeated with the filter described in (10). With $\tau = 0.01$, the systematic error increased from 4.52 m/s to 17.75 m/s, while the random error dropped from 1.5 m/s to 0.27 m/s (assuming $\sigma_f = 10$ Hz).

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Model Error and the Direction-Finder Problem

Abstract

The error introduced into many direction-finder (DF) algorithms by the use of projections is discussed. Upper bounds for this error are calculated and formulas for calculating these bounds are extended to the case of nonconformal transformations.

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