ELSEVIER

Signal Processing 41 (1995) 165–174

SIGNAL PROCESSING

Perfect periodic correlation sequences

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Received 9 August 1993; revised 20 December 1993 and 25 April 1994

Abstract

Sequences with the property of perfect periodic autocorrelation are used in various fields such as communication and radar systems. Various methods were introduced to construct such sequences. Most of these methods concentrated on binary or N-phase sequences. In this paper two methods to construct general phase sequences are introduced. The methods are based upon the properties of the Fourier transform of perfect periodic autocorrelation sequences. The sensitivity of these codes to errors is also elaborated.

Zusammenfassung

Folgen mit der Eigenschaft einer vollkommen periodischen Autokorrelation verwendet man in verschiedenen Gebieten wie bei Kommunikations- und Radarsystemen. Verschiedene Verfahren zur Konstruktion solcher Folgen wurden eingeführt. Die meisten solchen Techniken beziehen sich auf binäre oder N-Phasen-Sequenzen. In diesem Beitrag werden zwei Methoden zur Konstruktion von Sequenzen allgemeiner Phasen eingeführt. Die Grundlage der Verfahren bilden die Eigenschaften der Fouriertransformierten von Folgen mit perfekt periodischer Autokorrelation. Auch die Fehlerempfindlichkeit so entworfener Codes wird herausgearbeitet.

Résumé

On fait usage de séquences d'autocorrélation périodiques parfaites dans divers domaines tels que les télécommunications et le radar. Il existe plusieurs méthodes de formation de séquences de ce genre, mais le plupart sont limitées à des séquences binaires ou N-phases. Cet article introduit deux methodes de génération qui conviennent à la formation de séquences de phases générales. Les méthodes sont basées sur les caractéristiques de la transformation de Fourier des séquences d'autocorrélation parfaites. On traite aussi la sensibilité de la fonction d'autocorrélation aux erreurs de phase.

Keywords: Perfect periodic correlation sequence; Butler matrix; Eigenvalue method; Iteration method; Error sensitivity

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SSDI 0165-1684(94)00098-0

1. Introduction

Let $u(n), n = 0, \ldots, M-1$, denote a length M sequence of complex numbers. If the cyclic (periodic) autocorrelation of this sequence,

$$R_u(k) = \sum_{n=0}^{M-1} u(n)\bar{u}^*(n+k),$$

$$\text{with } \tilde{u}(n+k) = u[(n+k) \bmod M], \tag{1}$$

has the property that

$$R_u(k) = E_p \delta_{k \bmod M}, \tag{2}$$

where δ_k is the Krönecker delta, and E_p the energy in a single period, then this sequence is said to be a perfect cyclic autocorrelation (PCA) sequence.

Sequences with such a property have been known for a long time, the earliest of which were described by Frank in [4] and Chu [3], who introduced sequences which are a sampled version of either a linear frequency modulation signal or a stepped frequency signal. A generation method for bi-phase sequences has been recently described in [1] and [6]. In [9] PCA sequences with non-uniform amplitude, thus having a degraded efficiency, are listed. Some of the codes properties are given in [7] and [5], and a method which could generate a code out of two known ones is shown. N-phase sequences are presented in [2].

One of the important properties of a PCA sequence is the fact that its Fourier transform is of uniform magnitude (see [9]). Using this property, two alternative methods for PCA sequence generation are presented hereunder. These methods are designed to produce a uniform amplitude signal, but with no restrictions set on the phase sequence. Thus the goal of the generation method is to find a set of phases $\{\phi_n\}$, $n = 1, \ldots, M$, such that the sequence given by

$$u(n) = e^{j\phi_n} \tag{3}$$

is a PCA sequence.

2. The eigenvector method

The requirement of a uniform spectral density function leads to an equivalent requirement for a uniform absolute value of the DFT of the original sequence. Since we require that both the sequence and its DFT are uniform, it is natural to look for an eigenfunction of the Fourier operator as a candidate code (an 'eigencode').

The Fourier operator will be translated to its matrix representation (Butler matrix)

$$B_M = \frac{1}{\sqrt{M}} e^{-j2\pi(kl/M)}, \quad k, l = 0, \ldots, M-1. \tag{4}$$

The set of matrix eigenvalues, as seen in Appendix A, is

$$\{1, j^n\}|_{n=2,\ldots,M}. \tag{5}$$

Let v_i denote an eigenvector of B_M, which corresponds to the eigenvalue λ_i. It is straightforward that

$$B_M^2 v_i = B_M \lambda_i v_i = \lambda_i^2 v_i.$$

v_i is also an eigenvector of B_M^2 corresponding to the eigenvalue λ_i^2. Since an eigenvalue of B_M is one of the set $\{1, j, -1, -j\}$ the eigenvalues of B_M^2 are either $+1$, which corresponds to the eigenvalues $\{1, -1\}$ of B_M, or -1, which corresponds to the $\{j, -j\}$ set of B_M eigenvalues.

Using this relation it can be shown, with the structure of B_M^2, as shown in Appendix A, that for $\lambda_i = \pm j$

$$B_M^2 v_i = \begin{pmatrix} 1 & 0 \\ 0 & J_{N-1} \end{pmatrix} \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iM} \end{pmatrix} = - \begin{pmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{iM} \end{pmatrix},$$

which results in the following equation set:

$$\begin{aligned} &v_{i1} = -v_{i1} \Rightarrow v_{i1} = 0, \\ &v_{ik} = -v_{i,M-k+2} \quad \forall k = 2, \ldots, \left[\frac{M-1}{2}\right], \end{aligned} \tag{6}$$

where $[\cdot]$ denotes the integer part function.

The fact that the first term is inherently zero prohibits the use of those eigenvectors as eigencodes, as described in this paper, since for an eigencode all the eigenvectors which correspond to a certain eigenvalue must form a base that spans a subspace in which a vector with all its terms of equal magnitude can be found.

As for $\lambda_k = \pm 1$, then

$$B_M^2 v_i = \begin{pmatrix} 1 & 0 \\ 0 & J_{N-1} \end{pmatrix} \begin{pmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kM} \end{pmatrix} = \begin{pmatrix} v_{k1} \\ v_{k2} \\ \vdots \\ v_{kM} \end{pmatrix},$$

following

$$v_{km} = v_{k,N-m+2} \quad \forall m = 2, \ldots, \left[\frac{M-1}{2}\right]. \tag{7}$$

Namely the eigenvectors are symmetric around the $[(M-1)/2]$th term (except for the first term).

Finding the eigenvectors corresponding to eigenvalues 1 and -1 is quite a straightforward procedure. However, since those are, in general, multiple-order eigenvalues, a set of vectors will be found for each eigenvalue, probably none of them uniform in magnitude. Those vectors span the subspace of all eigenvectors corresponding to 1 (or -1). A linear combination of them is to be found, to produce a uniform magnitude vector, which is the required eigencode.

Let the required vector have the following phase sequence:

$$\boldsymbol{\theta} = \{0, \theta_2, \ldots, \theta_M\}, \tag{8}$$

where the first term was set, without loss of generality, to 0. Let the column vector $\boldsymbol{e}^{j\theta}$ represent a term-by-term exponential of $\boldsymbol{\theta}$:

$$\boldsymbol{e}^{j\theta} = \{1, e^{j\theta_2}, \ldots, e^{j\theta_M}\}. \tag{9}$$

Let the matrix V_l be the matrix containing all the eigenvectors corresponding to eigenvalue l, and the column vector $\boldsymbol{a}$ be the vector of the coefficients of the required linear combination of the vectors in V_l. Then, we have to solve the following equation set:

$$V_l \boldsymbol{a} = \boldsymbol{e}^{j\theta} \tag{10}$$

with the unknowns $\boldsymbol{a}$ and $\boldsymbol{\theta}$.

Due to the symmetry expressed in Eq. (7), the last $[(M-1)/2]$ equations are identical to the $[(M-1)/2]$ equations, from the second one down. The number of independent complex equations is then $N_{eq} = M - [(M-1)/2]$, which gives

$$N_{eq} = \begin{cases} \dfrac{M}{2} + 1 & \text{for } M \text{ even}, \\ \dfrac{M+1}{2} & \text{for } M \text{ odd}. \end{cases} \tag{11}$$

When counting unknowns, it should be noted that there are two types of unknowns: the phases and the vector coefficients. As for the number of unknown phases, N_θ, it is equal to the number of equations, except for θ_1, which have already been set to 0, namely

$$N_\theta = N_{eq} - 1. \tag{12}$$

The number of unknowns and the number of independent equations depend upon the sequence length, M, as summarized in Table 1.

The number of coefficients in $\boldsymbol{a}$, N_a, is dependent upon the multiplicity of the respective eigenvalues. According to Eq. (A.11), the multiplicity of the $+1$

Table 1
Number of equations (N_{eq}) and unknowns (N_u) for the $+1$ and -1 eigenvalues

M	N_a	N_θ	N_u	$2N_{eq}$
+1 eigenvalue				
0 mod 4	$\frac{M+4}{4}$	$\frac{M}{2}$	$M+2$	$M+2$
1 mod 4	$\frac{M+3}{4}$	$\frac{M-1}{2}$	$M+1$	$M+1$
2 mod 4	$\frac{M+2}{4}$	$\frac{M}{2}$	$M+1$	$M+2$
3 mod 4	$\frac{M+1}{4}$	$\frac{M-1}{2}$	M	$M+1$
−1 eigenvalue				
0 mod 4	$\frac{M}{4}$	$\frac{M}{2}$	M	$M+2$
1 mod 4	$\frac{M-1}{4}$	$\frac{M-1}{2}$	$M-1$	$M+1$
2 mod 4	$\frac{M+2}{4}$	$\frac{M}{2}$	$M+1$	$M+2$
3 mod 4	$\frac{M+1}{4}$	$\frac{M-1}{2}$	M	$M+1$

eigenvalue is given by $N_a = [(M + 4)/4]$, whereas the multiplicity of the -1 eigenvalue is given by $N_a = [(M + 2)/4]$.

Table 1 sums up the number of equations and unknowns for eigenvalues $+1$ and -1 and for various cases of M. The count is made for real variables and equations, so it would be possible to add together the number of real phases and the number of complex valued $\boldsymbol{a}$ vector coefficients.

As shown in the table, the number of unknowns is equal to the number of equations, only in 2 out of the 8 cases presented. Only in these cases is it possible to solve Eq. (10) analytically. In the other cases, the number of equations exceeds that of the unknowns, and solution is possible only in the least-squares sense. It should be noted however that there is no proof that such a solution does exist.

Table 3 in Appendix B presents some of the solutions found for various M's. The equations were solved numerically using the steepest gradient method. $M = 37$ was the longest code searched, although longer codes may probably exist. The program could not reach a solution for $M = 8, 9$ and 16.

It is interesting to see that the procedure described reached sometimes codes using an alphabet of only 2 (for $M = 4$), 3 (for $M = 5, 13$) and 5 (for $M = 17$) symbols (phases). Since the number of free phases is $[(M + 1)/2]$, it is quite obvious for low-order M's but for large M's it is not straightforward. There is no special reason for this phenomenon, except for the fact that such codes do exist.

3. The iteration method

Another approach is of a more heuristic nature. It is an iterative method. The procedure is described as follows:

(1) In the first iteration, take a random sequence of numbers on the unit circle in the complex plane:

$$u_0 = e^{j\theta_0(n)}, \quad \{\theta_0(n)\}|_{n=0,\ldots,M-1}.$$

(2) In the ith iteration, perform its discrete Fourier transform, and present it in its polar form:

$$U_i(k) = F(u_i) = \sum_{0}^{M-1} u_i(n) e^{-j2\pi(nk/M)}$$

$$= \rho_i(k) e^{j\phi_i(k)}, \quad k = 0, \ldots, M-1.$$

(3) Check whether the resulting amplitude is uniform by calculating, for example, the standard deviation of the magnitude normalized to its average over k, $\bar{\rho}_i$, and comparing to a predefined limit, ε_ρ:

$$\underset{k}{\mathrm{STD}}\left[\frac{\rho_i(k)}{\bar{\rho}_i}\right] \leqslant \varepsilon_\rho.$$

If the condition is met, the algorithm is stopped.

(4) If the condition is not met, repeat steps 2–4, with the sequence

$$u_{i+1}(n) = e^{j\phi_i(n)}, \quad k = 0, \ldots, M-1.$$

This procedure is similar to a well-known method of extracting the eigenvalues and eigenvectors of a matrix, but here the algorithm tries to force a uniform amplitude solution. Empirically it was found that the procedure converges, but with a constantly decreasing rate. In this method the exact code is not reached and in this sense those codes are called here NPCA for nearly perfect cyclic autocorrelation.

A search was performed for various values of M. For each M it was possible to find several sequences. The criterion for stopping the iterations was taken as $\varepsilon_\rho = 2 \times 10^{-4}$; however, the procedure stopped after 1000 iterations in case the limit was not reached. Table 4 in Appendix B shows a sample code for each M that was searched, with the standard deviation of the spectrum of the code.

4. Error sensitivity

The actual sidelobe level achieved, when a PCA based signal is used within a system, depends upon the accuracy in code generation. We shall not refer here to Doppler errors, which are analysed in [8]. A distinction shall be made here between code error sensitivity, which is the sensitivity of the autocorrelation sidelobe level to errors in the sequence terms,

and spectrum error sensitivity, which is the sensitivity of the sidelobe level to a deviation of the spectrum magnitude from unity. This sensitivity is used to set the limit for stopping the iterations in the algorithm previously described.

4.1. Code error sensitivity

The model used for the analysis of code error sensitivity is as follows: A code generator, designed for the sequence $u(n)$, $n = 0, \ldots, M$, generates an erroneous version of the sequence, $\tilde{u}(n)$, given by

$$\tilde{u}(n) = u(n)[1 + e(n)], \tag{13}$$

where $e(n)$ is a complex stochastic process. Both components of the error process $e(n)$, $e_r(n) = \mathrm{Re}\{e(n)\}$ and $e_i(n) = \mathrm{Im}\{e(n)\}$, are assumed to be stationary, white, independent zero mean processes, an assumption most commonly used. We shall further denote

$$\mathrm{E}\{e_r^2\} = \sigma_r^2,$$

$$\mathrm{E}\{e_i^2\} = \sigma_i^2,$$

$$\sigma_i^2 + \sigma_r^2 = \sigma^2.$$

On transmission, a single period of $\tilde{u}(n)$ is recorded and used to continuously modulate the transmitted signal. The received signal is correlated with that recorded period.

The resulting autocorrelation function is given by

$$\begin{aligned}\tilde{R}_u(k) &= \sum_{n=0}^{M-1} \tilde{u}(n)\bar{\tilde{u}}^*(n+k) \\ &= \sum_{n=0}^{M-1} u(n)\bar{u}^*(n+k)[1+e(n)] \\ &\quad \times [1 + e^*(n+k)].\end{aligned} \tag{14}$$

Assuming ergodicity, we shall compute the statistical ensemble expected value and variance in order to evaluate the mean and sidelobe level in the time domain.

The expected value of the autocorrelation function is then

$$\begin{aligned}&\mathrm{E}\{\tilde{R}_u(k)\} \\ &\quad = R_u(k) + \sum_{n=0}^{M-1} u(n)\bar{u}^*(n+k)[\sigma_r^2 + \sigma_i^2]\delta_k,\end{aligned}$$

$$E\{\tilde{R}_u(k)\} = \begin{cases} 0, & k \neq 0 \bmod M, \\ M(1+\sigma^2), & k = 0 \bmod M. \end{cases} \tag{15}$$

To compute the variance, one has to compute

$$\begin{aligned}&\mathrm{E}\{|\tilde{R}_u(k)|^2\} \\ &\quad = \mathrm{E}\Bigg\{\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} u(n)\bar{u}^*(n+k)u^*(m)\bar{u}(m+k) \\ &\qquad \times [1+e(n)][1+e^*(n+k)] \\ &\qquad \times [1+e^*(m)][1+e(m+k)]\Bigg\}.\end{aligned} \tag{16}$$

Some of the cross terms cannot be directly evaluated; however, a limit can be found to yield

$$\begin{aligned}&\mathrm{Var}\{|\bar{R}_u(k)|\} = \mathrm{E}\{|\tilde{R}_u(k)|^2\} - \mathrm{E}^2\{|\tilde{R}_u(k)|\} \\ &\leqslant \begin{cases} M\sigma^2(\sigma^2+2) + 2M|\sigma_r^2 - \sigma_i^2|, & k \neq 0 \bmod M, \\ 4M\sigma_r^2 + M(K-1)(\sigma_r^4 + \sigma_i^4), & k = 0 \bmod M, \end{cases}\end{aligned} \tag{17}$$

with K the fourth moment of the distribution of e_r and e_i. The third moment of that distribution was assumed to be zero.

The rms sidelobe level, which is the figure of merit for error sensitivity, is given by

$$\begin{aligned}\mathrm{SLL} &= \frac{1}{M^2}\mathrm{Var}\{|\tilde{R}_u(k)|\}|_{k\neq 0} \\ &\leqslant \frac{\sigma^2(\sigma^2+2) + 2|\sigma_r^2 - \sigma_i^2|}{M}.\end{aligned} \tag{18}$$

For the special case where the errors in both axes are small and of a similar order of magnitude, $\sigma_r \approx \sigma_i \ll 1$,

$$\mathrm{SLL} \leqslant \frac{\sigma^2(\sigma^2+2)}{M} \approx \frac{2\sigma^2}{M}. \tag{19}$$

Fig. 1 shows the RMS sidelobe level (in dB) as a function of phase error standard deviation (in degrees). The graph shown is a result of simulations

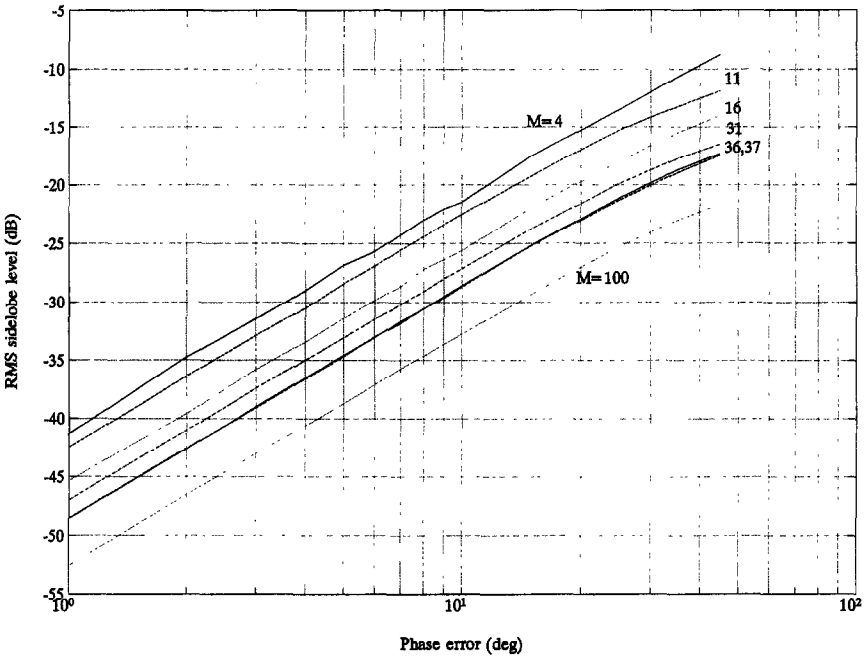

Fig. 1. RMS sidelobe level as a function of random phase error standard deviation for various sequence lengths.

(250 runs per data point) performed in adding random phase error to PCA sequences of various lengths. From the point of view of SLL, no difference can be observed between these codes and Chu codes [3], or shift register sequence codes [6].

Fig. 2 shows a comparison between the theoretical limit of Eq. (16) and simulation results of the autocorrelation of a PCA sequence of length 31. Fig. 2 shows as well the peak sidelobe level as a function of the phase error. The peak sidelobe is some 10 dB higher than the RMS level.

4.2. Spectrum magnitude error

In order to set the limit in step 2 of the iteration method described above, it is essential to determine the sidelobe level of the autocorrelation as a function of the deviation of the spectrum from unity. This figure of merit is also necessary in determining the requirements for spectral in-band uniformity of a system in which such a sequence is implemented.

Let the code found be $\{u(k) = e^{j\phi_k}\}|_{k=0,\dots,M-1}$. Its normalized spectrum magnitude $|U(l)|^2$ will be modelled by a constant plus an error, e_l:

$$|U(l)|^2 = 1 + e_l. \tag{20}$$

The error e_l will be assumed to be a white stochastic process with zero mean and standard deviation $-\sigma$ (which is eventually the limit, ε_ρ, set in the iteration process). In order to make the calculation simpler we shall further assume ergodicity and replace the time averages by the statistical expected value.

The autocorrelation of the code will be given by

$$R_u(n) = F^{-1}\{|U(l)|^2\} = \frac{1}{M}\sum_{l=0}^{M-1} |U(l)|^2 e^{j2\pi(nl/M)},$$

$$n = 0, \dots, M-1. \tag{21}$$

Its average value is

$$E\{R(n)\} = \frac{1}{M}\sum_{l=0}^{M-1} E\{|U_l|^2\} e^{j2\pi(nl/M)} = \delta_{n0}, \tag{22}$$

with δ the Krönecker delta.

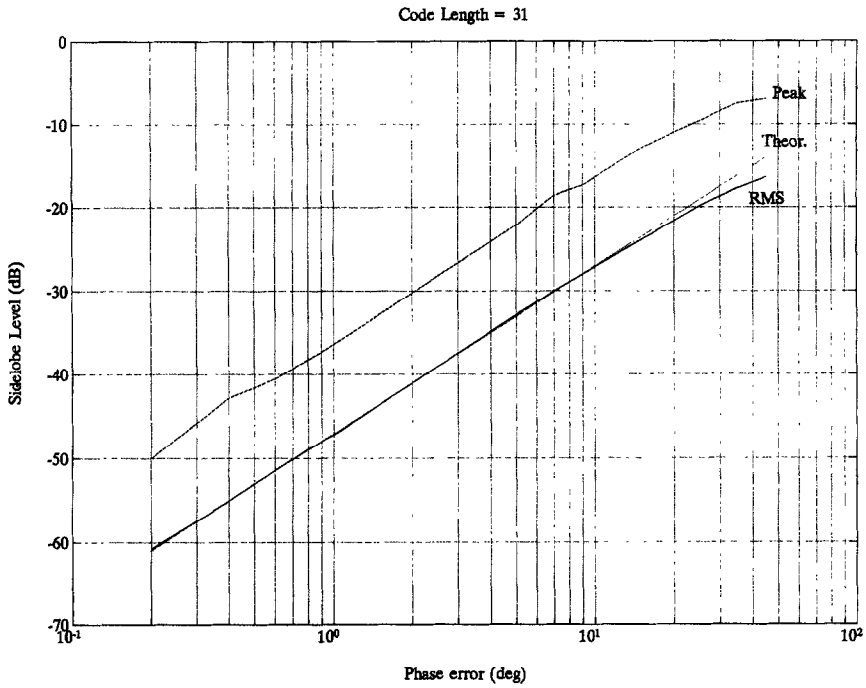

Fig. 2. Comparison between RMS sidelobe level, peak sidelobe level and the theoretical limit of a PCA code.

The variance of the autocorrelation function is

$$\begin{aligned}&\mathrm{E}\{[R(n)-\delta_{n0}][R^*(n)-\delta_{n0}]\}\\&=\frac{1}{M^2}\sum_{k=0}^{M-1}\sum_{l=0}^{M-1}(1+\sigma^2\delta_{lk})\mathrm{e}^{\mathrm{j}2\pi(l-k)n/M}-\delta_{n0}\\&=\frac{\sigma^2}{M}.\end{aligned}\tag{23}$$

The results of both code error sensitivity for random phase errors and the spectrum error sensitivity are quite intuitive. As the sidelobe level at a particular point is a sum of M almost independent variables, each with standard deviation σ, it gives, in first-order approximation, a sidelobe of $\sigma/\sqrt{M}$.

5. Conclusion

New methods for the generation of PCA sequences were introduced. These methods produce multiphase sequences for virtually any required length. Some example sequences are presented as well. The PCA code sensitivity to errors (both code error and spectrum error) is also studied.

Appendix A. Properties of the Butler matrix

A.1. Definition

The Butler matrix appears widely within the context of signal analysis. The Butler matrix of order M is defined as

$$B_M(i,k)=\frac{1}{\sqrt{M}}\exp\left(-\mathrm{j}\frac{2\pi ik}{M}\right)=\frac{1}{\sqrt{M}}W_M^{ik},$$

where

$$W_M=\mathrm{e}^{-\mathrm{j}(2\pi/M)},\quad i,k=\{0,\ldots,M-1\}.\tag{A.1}$$

Some of its properties will be given hereunder.

A.2. General properties

By definition the Butler matrix is a symmetric matrix. It can also be shown that it is unitary:

$$B_M B_M^*(i,k) = \frac{1}{M}\sum_{m=0}^{M-1} W_M^{im} W_M^{-mk} = \frac{1}{M}\sum_{m=0}^{M-1} \mathrm{e}^{-\mathrm{j}(2\pi/M)m(i-k)} = \delta_{ik},$$

with δ_{ik} is the Krönecker delta, and hence

$$B_M B_M^* = I_M, \tag{A.2}$$

where I_M is the unity matrix of dimension $M \times M$. This property is another representation of the fact that the inverse Fourier transform is the conjugate of the forward Fourier transform.

A.3. The powers of the Butler matrix

The powers of B_M are

$$B_M^2(i,k) = \frac{1}{M}\sum_{m=0}^{M-1} W_M^{im} W_M^{mk} = \frac{1}{M}\sum_{m=0}^{M-1} \mathrm{e}^{-\mathrm{j}(2\pi/M)m(i+k)} = \begin{cases} 1 & \text{if } i+k=0 \text{ or } i+k=M, \\ 0 & \text{elsewhere.} \end{cases} \tag{A.3}$$

The resulting matrix has the following structure:

$$B_M^2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}_{M-1}^{\mathrm{T}} \\ \mathbf{0}_{M-1} & J_{M-1} \end{pmatrix},$$

where J_{M-1} is the $(M-1)\times(M-1)$ matrix with the secondary diagonal all ones and $\mathbf{0}_{M-1}$ denotes an all-zero column vector of size $M-1$. This property represents the fact that applying the Fourier transform twice leads to a time reversal operation.

With a similar block decomposition applied to the B_M matrix, and $\mathbf{1}_{M-1}$ denotes an all-one vector, one can write

$$B_M^3 = \begin{pmatrix} 1 & \mathbf{0}_{M-1}^{\mathrm{T}} \\ \mathbf{0}_{M-1} & J_{M-1} \end{pmatrix}\begin{pmatrix} 1 & \mathbf{1}_{M-1}^{\mathrm{T}} \\ \mathbf{1}_{M-1} & W_M^{ik}|_{i,k=1}^{M-1} \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{1}_{M-1}^{\mathrm{T}} \\ 1_{M-1} & W_M^{(M-i)k}|_{i,k=1}^{M-1} \end{pmatrix} = W_M^{-ik}|_{i,k=1}^{M-1} = B_M^{-1}, \tag{A.4}$$

yielding directly

$$B_M^4 = B_M^{-1} B_M = I_M. \tag{A.5}$$

A.4. Butler matrix eigenvalues

Let $f(\lambda)$ denote the polynomial

$$f(\lambda) = \lambda^4 - 1. \tag{A.6}$$

From (A.5), it is clear that $f(B_M) = 0$; thus the eigenvalues of B_M are contained within the set of the roots of $f(\lambda)$. Namely the eigenvalues of B_M are $\{+1, -1, +\mathrm{j}, -\mathrm{j}\}$ with multiplicities $\{m_1, m_{-1}, m_{\mathrm{j}}, m_{-\mathrm{j}}\}$, respectively. To find the multiplicities the following two properties shall be used:

(1) The trace of the Butler matrix, which is the sum of the eigenvalues, is given by the Gauss sums [10], written here in a more compact form:

$$\sum_{i=0}^{M-1} \lambda_i = \mathrm{tr}\{B_M\} = \frac{1}{\sqrt{M}}\sum_{n=0}^{M-1} \mathrm{e}^{-\mathrm{j}(2\pi/M)n^2} = \begin{cases} 1-\mathrm{j}, & M \equiv 0 \bmod 4 \\ 1, & M \equiv 1 \bmod 4 \\ 0, & M \equiv 2 \bmod 4 \\ -\mathrm{j}, & M \equiv 3 \bmod 4 \end{cases} = \frac{1+\mathrm{j}^M}{1+\mathrm{j}}. \tag{A.7}$$

(2) If λ_i is an eigenvalue of B_M, then λ_i^2 is an eigenvalue of B_M^2, see Eq. (6) above. The eigenvalues of B_M^2 are then $\{1, -1\}$, with multiplicities $(m_1 + m_{-1}, m_{\mathrm{j}} + m_{-\mathrm{j}})$, respectively. The eigenvalues of B_M^2 can be directly calculated, by finding the roots of the characteristic polynomial of B_M^2:

$$|\lambda I_M - B_M^2| = 0. \tag{A.8}$$

Using (A.3), it can be shown that by subsequent decomposition, the determination on the left-hand

side of Eq. (A.8) can be reduced to

$$|\lambda I_M - B_M^2| = (\lambda - 1)|\lambda I_{M-1} - J_{M-1}|$$
$$= \begin{cases} (\lambda - 1)^2(\lambda^2 - 1)^{M/2-1}, & M \text{ even}, \\ (\lambda - 1)(\lambda^2 - 1)^{(M-1)/2}, & M \text{ odd}, \end{cases}$$

and the equations to be solved:

$$\begin{cases} (\lambda - 1)^{M/2+1}(\lambda + 1)^{M/2-1} = 0, & M \text{ even}, \\ (\lambda - 1)^{(M+1)/2}(\lambda + 1)^{(M-1)/2} = 0, & M \text{ odd}. \end{cases} \quad \text{(A.9)}$$

Thus the multiplicity values $\{m_i\}|_{i=\{\pm 1, \pm j\}}$ are constrained by the following:

$$\text{(i)} \quad m_1 - m_{-1} = \begin{cases} 1, & M \equiv 0 \bmod 4, \\ 1, & M \equiv 1 \bmod 4, \\ 0, & M \equiv 2 \bmod 4, \\ 0, & M \equiv 3 \bmod 4, \end{cases}$$

$$\text{(ii)} \quad m_{-j} - m_j = \begin{cases} 1, & M \equiv 0 \bmod 4, \\ 0, & M \equiv 1 \bmod 4, \\ 0, & M \equiv 2 \bmod 4, \\ 1, & M \equiv 3 \bmod 4, \end{cases} \quad \text{(A.10)}$$

$$\text{(iii)} \quad m_1 + m_{-1} = \left\lceil \frac{M}{2} \right\rceil + 1,$$

$$\text{(iv)} \quad m_j + m_{-j} = \left\lceil \frac{M}{2} \right\rceil - 1.$$

Equations (i) and (ii) in (A.10) are the result of breaking Eq. (A.7) into its real and imaginary part. Equations (iii) and (iv) are the direct result of (A.9).

The solution of (A.10) is straightforward:

$$m_1 = \left\lceil \frac{M+4}{4} \right\rceil, \quad m_{-1} = \left\lceil \frac{M+2}{4} \right\rceil,$$
$$m_j = \left\lceil \frac{M-1}{4} \right\rceil, \quad m_{-j} = \left\lceil \frac{M+1}{4} \right\rceil. \quad \text{(A.11)}$$

it follows directly from Eq. (A.11) that the set of eigenvalues of the Butler matrix of order M is given by

$$\{1, j^k\}|_{k=2,\ldots,M}. \quad \text{(A.12)}$$

Table 2 shows the eigenvalues of Butler matrices of orders 1 to 6 and demonstrates (A.12).

Table 2
List of eigenvalues for various-order Butler matrices

M	Eigenvalues
1	1
2	1, −1
3	1, −1, −j
4	1, −1, −j, 1
5	1, −1, −j, 1, j
6	1, −1, −j, 1, j, −1

Appendix B

Table 3
List of eigencodes

M	ϕ_k (degrees)
4	0, 0, 180, 0
5	0, −72, 72, 72, −72
12(1)	0, −30, −60, 90, 120, −30, 180, −30, 120, 90, −60, −30
(2)	0, 30.3, 61.2, −97.6, −76.3, 134.5, −15.1, 134.5, −76.3, −97.6, 61.2, 30.3
13	0, 77.5, −77.5, 77.5, 77.5, −77.5, −77.5, −77.5, −77.5, 77.5, 77.5, −77.5, 77.5
17	0, 121.2, −121.2, 24.7, 121.2, 24.7, −24.7, −24.7, −121.2, −121.2, −24.7, −24.7, 24.7, 121.2, 24.7, −121.2, 121.2
37	0, −77.0, −32.7, −113.4, −46.4, 75.1, 145.2, 4.0, 102.9, −112.0, 36.7, −62.0, 176.2, −165.4, 84.8, 70.3, −0.3, −68.2, 76.1, 76.1, −68.2, 0.3, 70.3, 84.8, −165.4, 176.2, −62.0, 36.7, −112.0, 102.9, 4.0, 145.2, 75.1, −46.4, −113.4, −32.7, −77.0

Table 4
List of NPCA codes

M	ϕ_k (degrees)	ε_p
4	0, 225.03, 0.01, 45.02	1.670×10^{-4}
5	0, 72, 72.01, 0.01, 216.01	1.386×10^{-4}
6	0, 209.99, 120, 90.01, 120.01, 210.01	1.574×10^{-4}
7	0, 154.3, 87.2, 102.9, 35.7, 190, 15.7	1.955×10^{-4}
8	0, −4.9, 170.1, 19.5, 80.1, 175.1, 90, −29.4	1.490×10^{-4}
9	0, 31.6, 27.6, 200, 83, 243, −41, 107.6, 71.6	1.764×10^{-4}
16	0, 154.8, −175.4, −31.4, 179.4, 99.4, 132.7, −167.7, 125.7, −136.3, 3.8, −112.2, 106.5, 60.5, 99.0, −20.6	9.8×10^{-5}
31	0, −124.6, −35.8, −206.6, −212.6, 72.4, −105.5, 120.4, 59.4, −124.8, −95.2, −100.3, −182.7, −164.6, 1.82, −81.7, −124.4, −55.4, −218.9, −150.4, −133.8, 14.2, −100.5, 86.9, 102.2, 102.0, −229.2, −206.2, −61.7, 15.8, −179.2	9.96×10^{-5}
50	0, 163.3, 29.2, 4.6, 2.2, 258, 241.4, 252.3, 172.3, 91.4, 147.5, −50.6, 282, 90.9, −54.7, 70.2, 78, 91.1, 140.9, 172.4, −54.8, 154.7, 131.6, −18.1, 273.8, 217.1, 300.5, 70, 176.7, −39.8, 233.5, 37, 108.6, 212.6, 49.6, 199.3, −15.1, 33.4, 38.9, 91.1, −29.2, 122.8, 24.4, 206, 39, 13.2, −8.7 121.3 127.8 58.5	4.6×10^{-3}
100	0, 48.1, 255.3, 16.9, −17.3, −4.1, 119.4, 66.3, 86.5, 300.3, 268.1, −32.2, 28.8, 305.7, −33.2, 137.7, 249.9, 107.2, 194.5, 284.3, −33.1, 13.7, 189.8, 288.4, 71.5, 279.1, 95.5, 46.5, 217.1, 267.6, 306.9, 128.8, 36.4, 206.4, 69.5, 73.1, −24.8, 231.3, −2.3, 199.7, 225.2, 190.5, 210.4, 169.4, 158.8, −17.20, 304.2, 140.7, 66, 291.2, 127.2, 229, 148.1, 94.9, 183, 311, 226.2, 154.9, 238.9, −36.3, 16.4, 290.1, 158.6, 3.1, 0.5, 105, 116.8, −42.3, 291.4, 308.2, −11.6, 301.7, 254.5, 192.2, 200.1, −15, 241.1, 53.2, 299.4, 158.2, 100.7, 152.7, 47.4, 4, −11.6, 146.2, 202.5, 55.7, 95.9, 295, 160.6, 239.8, 275.6, 46.5, 193.5, −8.2, 160.9 −29.4, 252.6	3.60×10^{-3}
128	0, −79.2, −161.7, 65.7, −142.6, −115.5, −112.7, −150.7, −56.9, 113.2, 24.2, −28.5, −21.2, −12.6, 162.4, −122.6, −82.6, −11.8, −147.8, −55.0, −35.2, 12.3, −129.8, −157.8, −93.9, 6.7, 78.0, −166.2, 164.4, −47.8, −52.5, 3.9, 3.3, 35.7, −49.4, −4.1 −178.7, 82.0, 142.9, 145.2, 104.4, −78.3, 14.7, 109.4, −92.9, −162.5, −166.0, −174.1, −37.9, −176.9, 83.1, −56.4, −178.9, 109.6, −162.0, −24.3, −94.4, −72.0, 118.0, 93.2, −79.8, −175.1, −75.4, 23.1, 96.4, 145.1, 36.2, 93.1, 112.0, −42.6, −72.6, 86.6, −24.4, 35.3, −77.7, −68.8, 168.0, −18.2, 103.3, 3.0, −82.4, −48.5, −127.4, 68.9, 174.1, −116.7, 104.1, −146.0, −160.3, 93.2, −98.6, 153.1, 4.5, 136.0, 60.4, 133.0, −90.1, −15.1, −144.8, −117.6, 85.6, −15.4, 167.4, 174.1, 37.3, −14.0, −28.3, −121.0, 112.9, 5.7, 147.5, 25.7, 95.0, −117.7, 124.8, −83.7, −178.5, −103.0, 84.7, −150.0, −8.2, −3.2, −134.9, −90.9, 41.4, −40.6, −39.3, 11.7	1.23×10^{-2}

Notes: (1) For small values of M, if the angles found are rounded, the code obtained will be a perfect PCA. This fact is not so obvious for large M. (2) Such a search was performed for large values of M, up to 1024.

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