The initial estimate of s_1 and s_2 is obtained by the Tufts–Kumaresan (TK) method [2], [6], [7], [11], that is by forward or backward linear prediction (depending on whether one is looking for falling or rising exponential components) after low-rank approximation to the observed data matrix or an estimated correlation matrix. A standard quasi-Newton method is then used for the maximization of E_1 [8]. The performance of this method is studied by computing the bias and the standard deviation of α_k's and s_k's obtained from 500 trials. A different realization of noise is used for each trial. Comparisons between the Newton estimates, the TK estimates, and the CR bound for unbiased estimates are presented in Fig. 1. The Newton estimates are almost unbiased and attain the standard deviation given by the CR bound. The TK estimates are biased and have variances that significantly differ from the CR bound. The improvements over the TK initial estimates are most significant at low SNR and when the two frequencies are closely spaced (case 1). For case 1 at 25 dB SNR, the mean-square error E for the TK method is about twice that for the Newton method.

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Any Two $N \times N$ Costas Signals Must Have at Least One Common Ambiguity Sidelobe if $N > 3$—A Proof

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Costas signals have an ambiguity function that approaches the ideal "Thumbtack" configuration. To increase the main lobe/sidelobe ratio without increasing the number of frequencies (signal dimension, N) it may be possible to stagger Costas signal pulses whose ambiguity sidelobe patterns do not coincide. We prove here that Costas signals of the same dimension, with completely different ambiguity sidelobe patterns, do not exist for N > 3.

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John P. Costas has suggested [1] a class of sonar and radar signals, whose range-doppler sidelobe peaks are well controlled to approach the ideal "Thumbtack" ambiguity function. A Costas signal belongs to a family of signals which can be represented by N dots in an otherwise null $N \times N$ matrix such that each row contains a single dot as does each column. If the columns represent consecutive time slots of duration dt, and the rows—distinct frequencies at equal spacing df, then such a representation implies that at any time slot only one frequency will be transmitted, and there will be no repetition of frequencies. Costas particular signals have an additional quality: The placement of dots is such that for all possible horizontal and vertical shift combinations of the matrix relative to an unshifted itself, at most one pair of dots will coincide. The horizontal and vertical shifts correspond to delay and doppler coordinates, respectively, of the ambiguity function. The number of coincidences is related to the value of the ambiguity function. Thus a Costas signal, in principle, should have a main lobe of level N and sidelobes of level 1. Coherent processing actually yields a more complicated ambiguity function, with the sidelobe peak levels (particularly near the main lobe) higher than 1.

Costas has shown that for dimensions $N \geqslant 2$ there are several different signals that meet the requirements of no more than one coincidence. For example, when $N = 3$ there are 4 Costas signals and when $N = 10$ there are 2160 such signals. Since each signal can create a different pattern of sidelobe peak locations, a possible way to increase the main lobe/sidelobe ratio can be to stagger different signals of the same dimension, and to integrate the output noncoherently. Intuitively it can easily be seen that many ambiguity sidelobes of one signal will fit in-between sidelobes of the other signal. The interesting question is whether there could be found two, or more, Costas signals of the same dimension without any coincidence of their ambiguity function sidelobes. Exhaustive computer search for signals up to order $N = 11$ did not yield such pairs, except at $N = 2$ and 3, which prompted a search for a proof that such pairs do not exist when $N > 3$. The proof is given below.

As Costas pointed out, the location of the sidelobes is determined by the difference triangle. Such a triangle can be best explained with the help of an example. Let the Costas signal be represented by the 7×7 frequency–time matrix shown in Fig. 1.

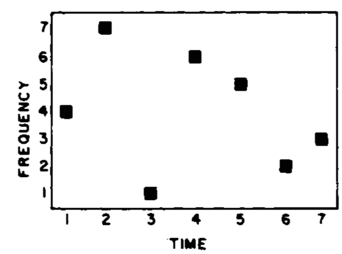

Fig. 1. Matrix representation of the Costas signal used in the example.

Such a signal can also be described by the sequence

$$\{a_i\} = 4,7,1,6,5,2,3. \tag{1}$$

The difference triangle of the signal described in Fig. 1 is given in Table 1. The first row of the difference triangle is formed by taking

Table 1 The Difference Triangle of the Sequence in (1)

(a_i)	4	7	1	6	5	2	3
$i = 1$	3	−6	5	−1	−3	1	
2	−3	−1	4	−4	−2		
3	2	−2	1	−3			
4	1	−5	2				
5	−2	−4					
6	−1						

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differences between adjacent terms in the sequence. The second row by taking differences between next-adjacent terms, and so on.

The value of the difference triangle in row i and column j is given by

$$D_{i,j} = a_{i+j} - a_j, \qquad i + j \leqslant N. \tag{2}$$

The relation between the difference triangle and the sidelobes pattern is such that for the delay difference given by i time slots there will be peaks at the doppler frequencies corresponding to the values of $D_{i,j}$. A Costas signal should have no repetition of values in any row of the difference triangle. The pattern of sidelobe peak locations for our example is given in Fig. 2.

DOPPLER FREQUENCY, D

DELAY, i	-6	-5	-4	-3	-2	-1	1	2	3	4	5	6
1	x			x		x	x		x		x	
2			x	x	x	x				x		
3				x	x		x	x				
4		x					x	x				
5			x		x							
6						x						

Fig. 2. A matrix representing sidelobe locations of the ambiguity function.

It should be noted that the sidelobe pattern in Fig. 2 represents only one half of the ambiguity function (positive delays). The second half is symmetrical with respect to the origin.

The matrix of sidelobe locations must obey the following rules:

a) Row i has $N - i$ sidelobes.
b) In any pair of columns $j, -j$ there are together $N - j$ sidelobes.
c) The matrix dimensions are $(N - 1) \times 2(N - 1)$.

Rule a) results from the fact that in a sequence of length N there is only one spacing of $N - 1$ (the spacing between the first and the last terms in the sequence), two spacings of $N - 2$ terms, etc.

Rule b) is due to the fact that the sequence is constructed from consecutive numbers. A difference D such that $|D| = N - 1$ must appear once, a difference of $|D| = N - 2$ must appear twice, etc.

Proving by contradiction it will be assumed that there are two Costas signals with completely different sidelobe patterns. Because the two sidelobe patterns have no common sidelobe locations, they can both be plotted on one $(N - 1) \times 2(N - 1)$ matrix. The sidelobe locations of both signals will be marked on Fig. 3.

	-6	-5	-4	-3	-2	-1	1	2	3	4	5	6
1	o	o	o	o	o	o	o	o	o	o	o	o
2		o	o	o	o	o	o	o	o	o	o	
3			o	o	o	o	o	o	o	o		
4				o	o	o	o	o	o			
5					o	o	o	o				
6						o	o					

Fig. 3. Combined ambiguity sidelobe locations of two hypothetical Costas signals which have no sidelobe coincidence.

We will first construct columns -1 and 1. Following rule b), each signal should have $N - 1$ sidelobes in these two columns, and together $2(N - 1)$ sidelobes (since no two sidelobes can occupy the same location). This means that the two columns must be full. We now note that in the last row, following rule a), there has to be one sidelobe for each signal. Hence, the two sidelobes must be in columns -1 and 1 and the remaining elements of the last row must be empty.

In the next pair of columns, -2 and 2, there should be $2(N - 2)$ occupied locations (rule b). However, not in the last row which should remain empty. Hence columns -2 and 2 must be filled except for the last row. Similar arguments can be applied to all the other pairs of columns, which will lead to the conclusion that if two such Costas signals exist, their ambiguity sidelobes should fill the triangle marked by circles in Fig. 3.

We will now try to construct the Costas signals that should yield the combined pattern in Fig. 3. Rule a) and Fig. 3 imply that the two occupied locations in the last row ($i = 6$) are each contributed by a different signal. In order to have an ambiguity sidelobe at a normalized doppler of -1 or 1, at the maximum delay, the signal should have a difference of ± 1 between the first and last terms of its sequence. In other words, one of the two signals should be described by the sequence

$$\{a_j\} = m, \cdots, m + 1 \tag{3}$$

and the second signal by

$$\{b_j\} = k, \cdots, k - 1. \tag{4}$$

Consider now the row before the last ($i = 5$). Rule a) implies that each signal should contribute two of the four differences $\{-2, -1, 1, 2\}$. There are six different combinations in which the sidelobes could be divided between the two signals. Each combination can be generated by two different sequences for each signal. The six combinations are investigated in Table 2, with the conclusion that

Table 2 Summary of the Six Major Combinations of the Last Two Rows of the Combined Ambiguity Sidelobe Pattern

Case No.	Last Rows in Fig. 3 (−2 −1 1 2)	Sequences that Each Can Yield the Pattern Indicated by the ×'s	Comment No.
1	O O × ×	$m, m-1, \cdots, m+1, m+1$	I
	O ×	$m, m \quad , \cdots, m+2, m+1$	I
2	O × O ×	$m, m-1, \cdots, m-1, m+1$	II
	O ×	$m, m+2, \cdots, m+2, m+1$	II
3	× O O ×	$m, m-1, \cdots, m-2, m+1$	III
	O ×	$m, m+3, \cdots, m+2, m+1$	III
4	O × × O	$m, m \quad , \cdots, m-1, m+1$	I
	O ×	$m, m+2, \cdots, m+1, m+1$	I
5	× O × O	$m, m \quad , \cdots, m-2, m+1$	I
	O ×	$m, m+3, \cdots, m+1, m+1$	I
6	× × O O	$m, m+2, \cdots, m-2, m+1$	IV
	O ×	$m, m+3, \cdots, m-1, m+1$	IV

Comments:

I. The sequence is not Costas because of two identical terms.
II. The sequence is not Costas because of two identical terms. However, at $N = 3$ the two identical terms (the second and the one before the last) converge to one, which makes it Costas.
III. This sequence is Costas but the sequences that can generate the pattern of O's are not Costas since the pattern is dual to case 4.
IV. There is a difference of 4 between the second term and the one before the last. Such a difference is not allowed in row $i = N - 3$. (In our example this corresponds to location $i = 4$, $j = 4$ in Fig. 3, which should be empty.)

both sequences, for at least one of the two signals necessary to generate each of the six combinations, cannot be a Costas sequence, when $N > 3$. Hence, we have proved that it is impossible to construct two Costas signals, of the same dimension N, with $N > 3$, which have completely different sidelobe patterns of their ambiguity function.

Note added on January 7, 1985: Solomon W. Golomb has called our attention to a similar proof by Herbert Taylor, in "Non-attacking rooks with distinct differences," University of Southern California, Communication Sciences Institute, Tech. Rep. CSI-84-03-02 (Mar. 1984).

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