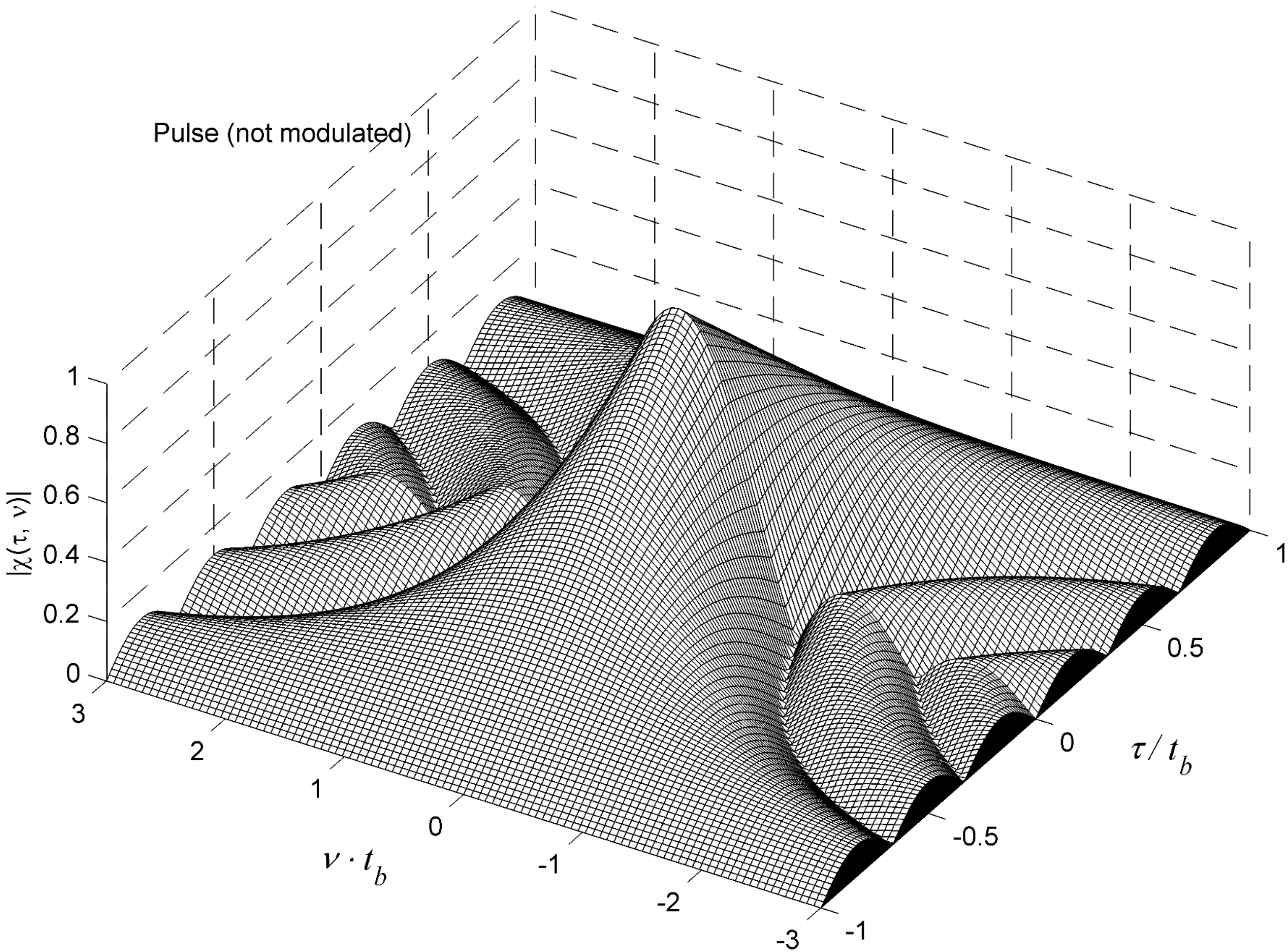
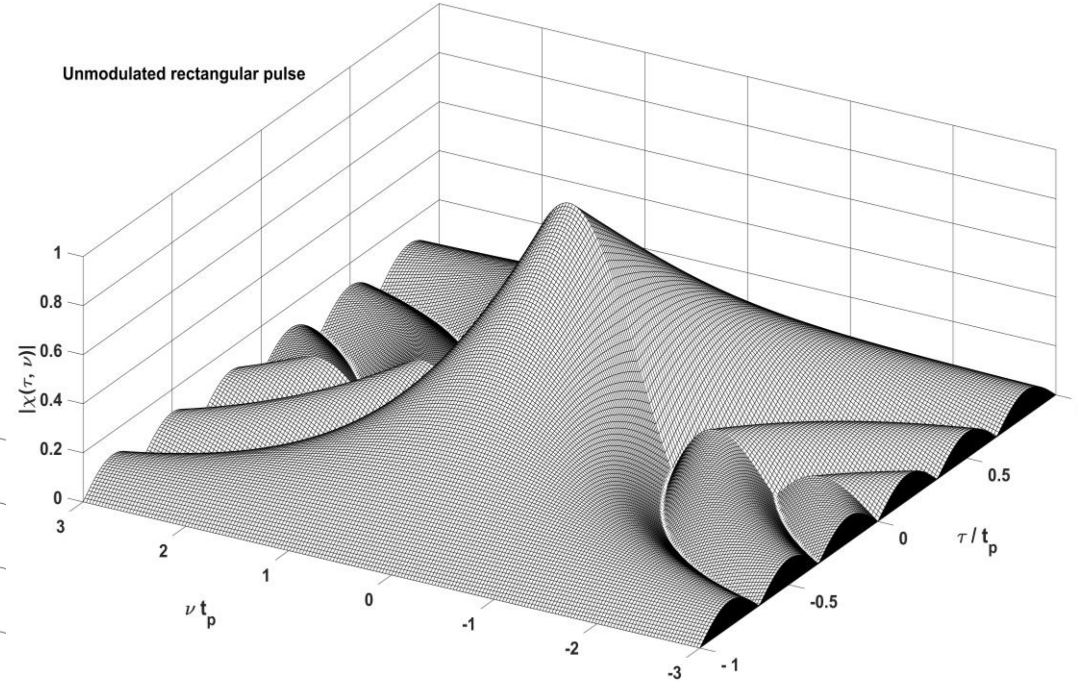
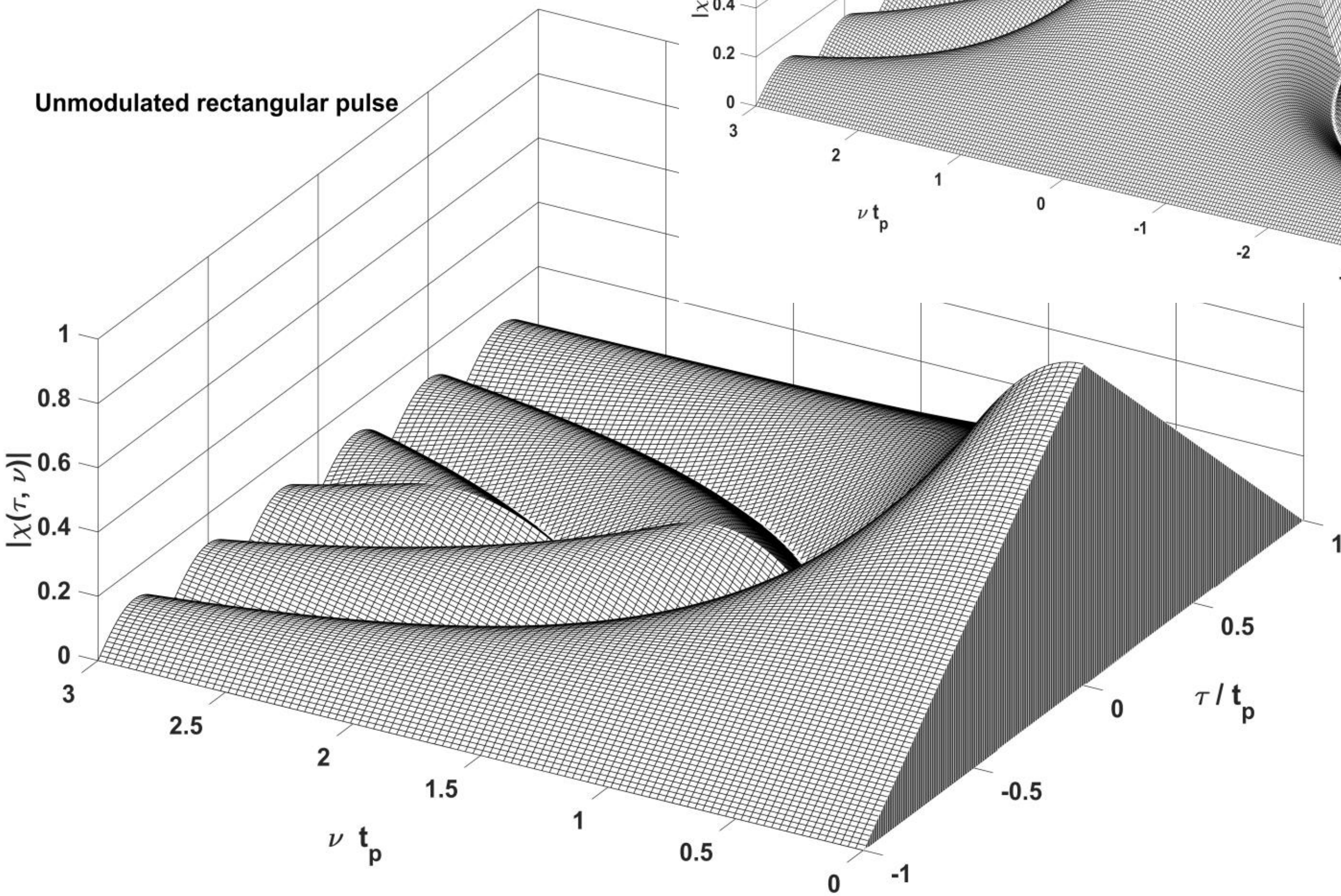
THE AMBIGUITY FUNCTION

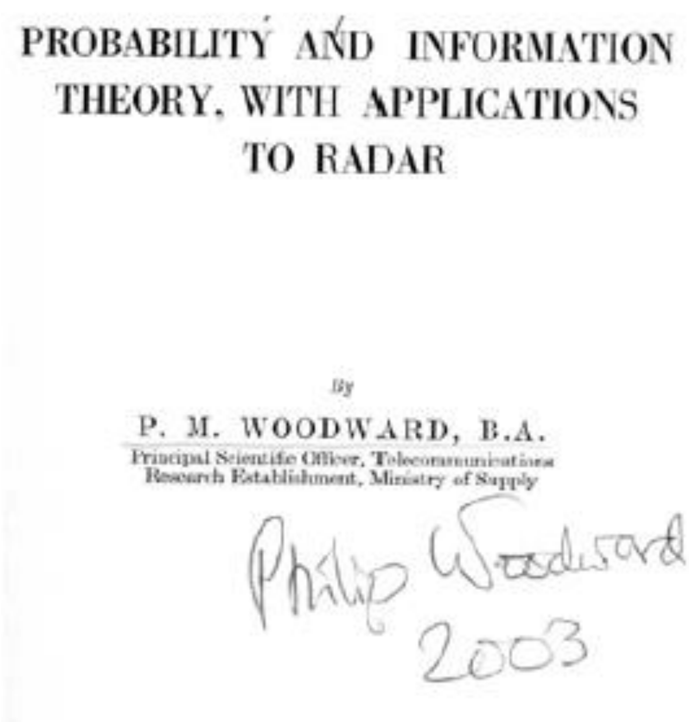


The AF is symmetric with respect to the origin.
Hence it is enough to display only 2 quadrants.
We will choose the positive Doppler quadrants.





P.M.Woodward:
Probability and information theory, with applications to radar (1953).


PROBABILITY AND INFORMATION THEORY, WITH APPLICATIONS TO RADAR

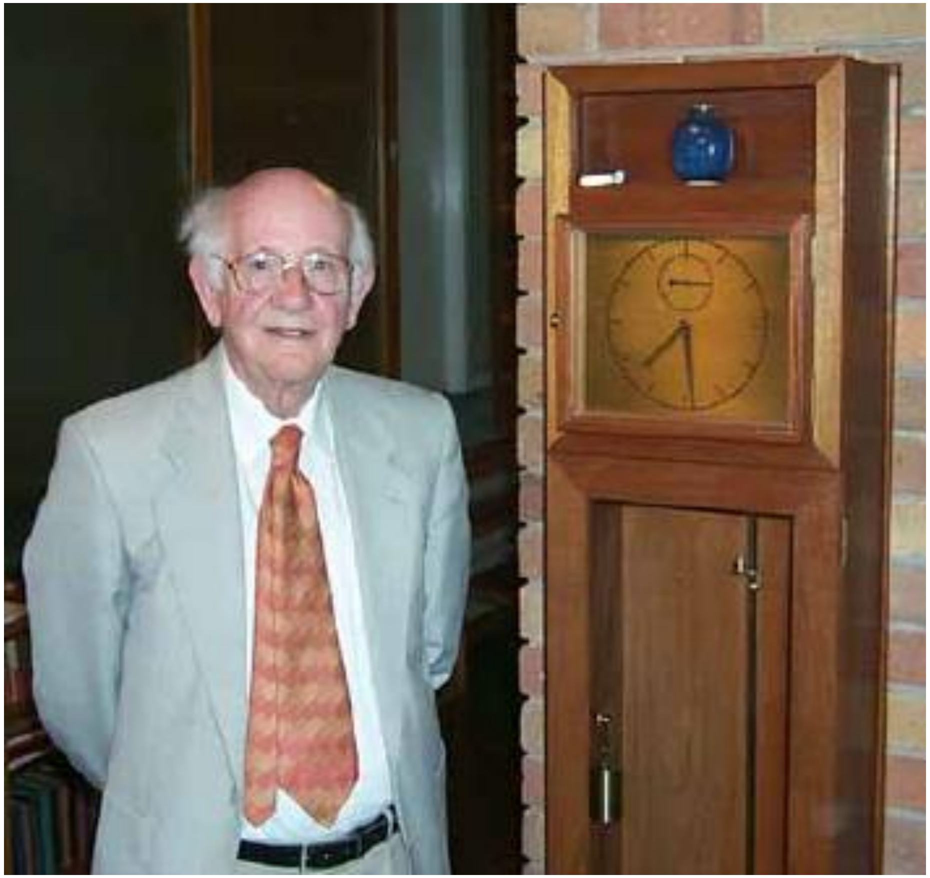
By

P. M. WOODWARD, B.A.

Principal Scientific Officer, Telecommunications Research Establishment, Ministry of Supply

Philip Woodward 2003

Courtesy of Lars Falk



2003

Courtesy of Lars Falk

Phillip Mayne Woodward
6.9.1919 – 30.1.2018

Phillip M. Woodward receiving the IEEE Picard medal, 2009

IEEE Dennis J. Picard Medal for Radar Technologies & Applications Recipients for 2009

PHILIP M. WOODWARD

Retired Deputy Chief Scientific Officer
Royal Radar Establishment
Malvern, United Kingdom

"For pioneering work of fundamental importance in radar waveform design, including the Woodward Ambiguity Function, the standard tool for waveform and matched filter analysis."

Philip Woodward

Philip Woodward has profoundly influenced radar signal analysis through his application of probability and statistics to recovering data from noisy samples. Dr. Woodward focused on optimizing the information content of the radar signal instead of its electrical strength in a time when the focus was on maximizing the electrical strength by comparison with that of the background noise.

He applied Bayesian probability techniques to eliminate everything but the desired information from radar echoes. The Woodward Ambiguity Function provided the foundation for the development of complex waveforms in modern radars and for description of radar resolution and accuracy. It was able to show graphically how range and velocity accuracy could be traded, how spurious responses appear in both dimensions and the limitations governing the process. With computing power not available at the time it was developed, it now has enabled system designers to assess the capacity of a complex radar transmission to detect the range and radial velocity of a target and to define the optimum detection strategy.

Dr. Woodward’s book, Probability and Information Theory, with Applications to Radar, is considered a classic in the field of radar, and his book entitled My Own Right Time is a classic in the field of horological science. With both fields in mind, the U.K. Royal Academy of Engineering awarded him their first-ever Lifetime Achievement Medal. Dr. Woodward retired in 1980 as a deputy chief scientific officer from the Royal Radar Establishment, where he began working in 1940. He currently resides in Malvern, United Kingdom.

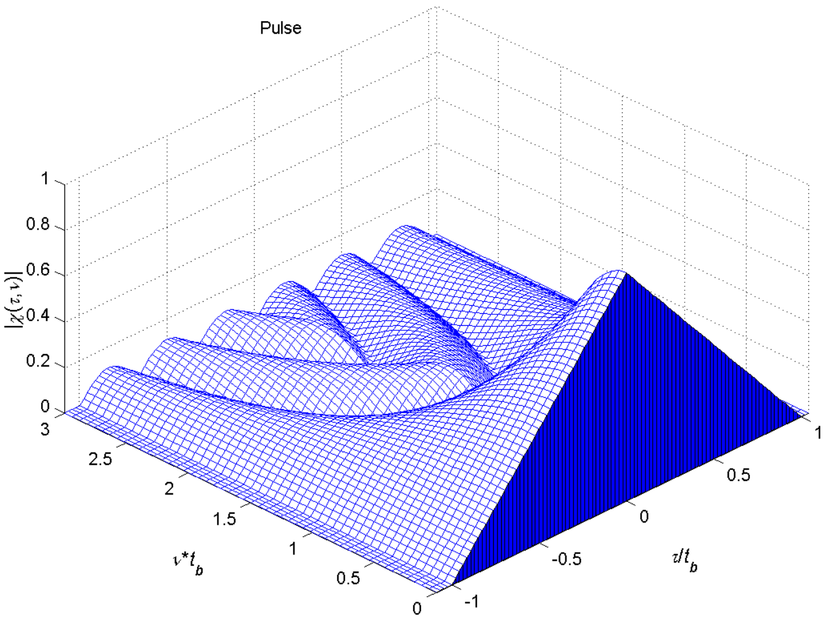
Nadav Levanon, Tel-Aviv University

$$\left|\chi(\tau,\nu)\right| = \left| \int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp(j2\pi\nu t)\,dt \right|$$

The ambiguity function represents the time response of a filter **matched** to a given **finite** signal, when that signal is received with a delay τ and a Doppler shift ν.

The ambiguity function is a two-dimensional function, where the two axes are τ and ν.

The ambiguity function depends only on the **complex envelope,** ***u(t),*** of the signal.



FOUR PROPERTIES of the AMBIGUITY FUNCTION

When the energy of $u(t)$ is equal to 1:

1. **Maximum at (0,0)**

$$|\chi(\tau,\upsilon)| \le |\chi(0,0)| = 1$$

2. **Constant volume**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\chi(\tau,\upsilon)|^2 d\tau d\upsilon = 1$$

For all signals (any energy):

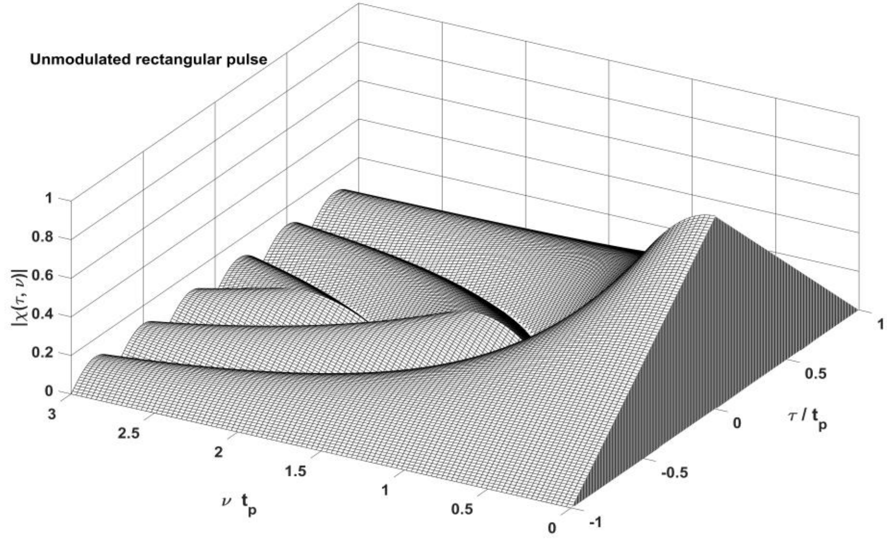
3. **Symmetry with respect to the origin**

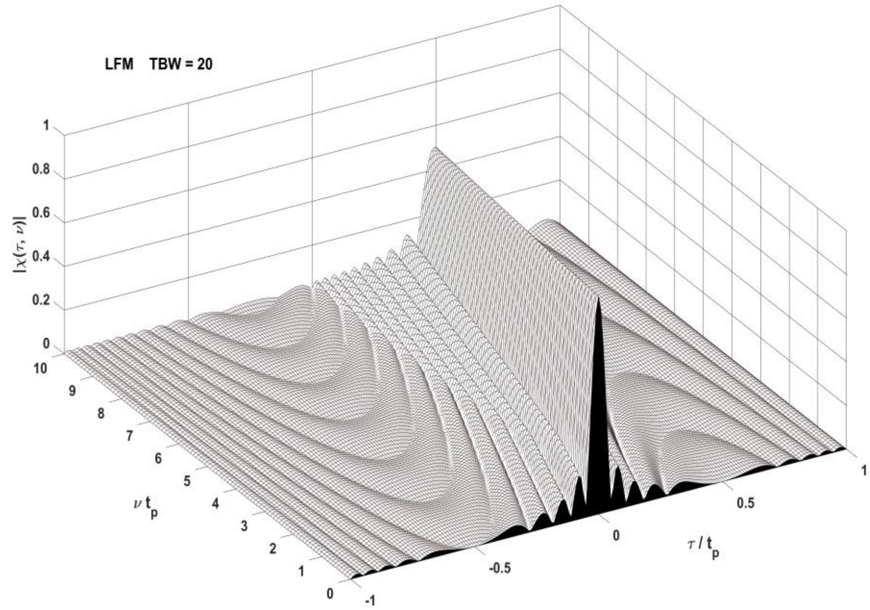
$$|\chi(-\tau,-\upsilon)| = |\chi(\tau,\upsilon)|$$

4. **Linear FM effect**

$$u(t) \Leftrightarrow |\chi(\tau,\upsilon)|$$

$$u(t)\exp\left(j\pi k t^2\right) \Leftrightarrow |\chi(\tau,\upsilon - k\tau)|$$





Proofs of the four AF properties can be found in Papoulis, A., *"Signal Analysis"*, McGraw Hill, 1977, Ch. 8.4.

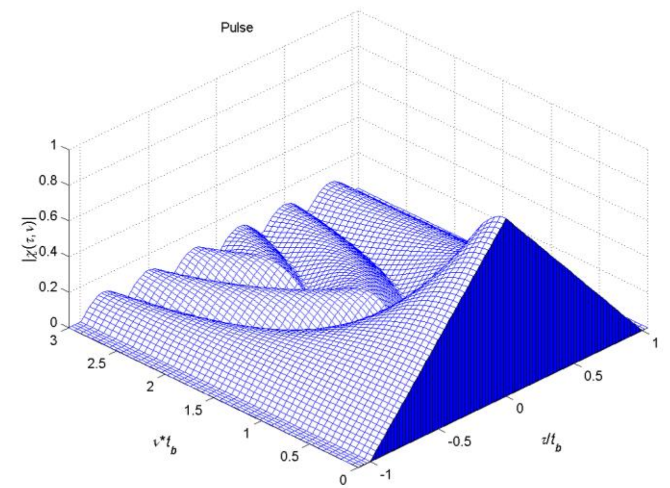
CUTS

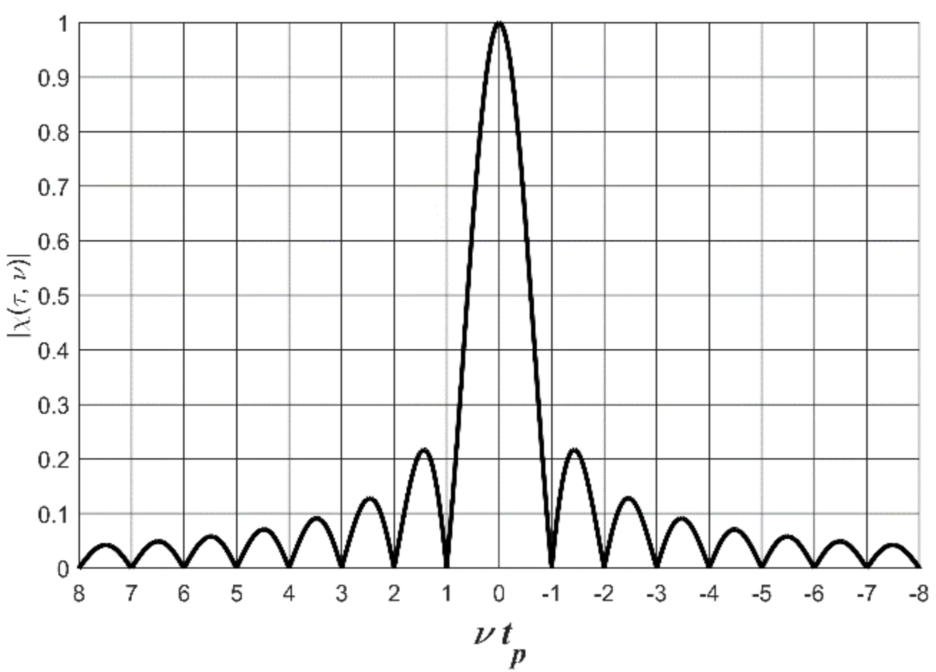
$$\left|\chi(\tau,0)\right| = \left|\int_{-\infty}^{\infty} u(t)u^*(t+\tau)dt\right| = \left|R(\tau)\right|$$

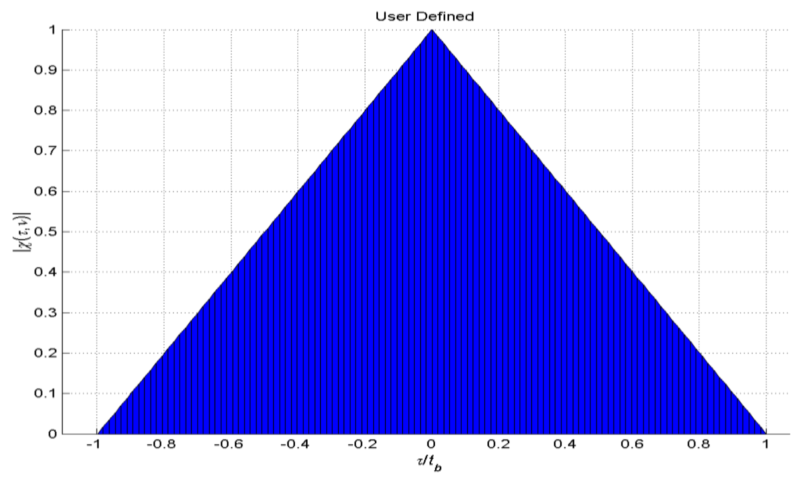
i.e., the cut at zero Doppler is the **autocorrelation** function of *u(t)*.

$$\left|\chi(0,\upsilon)\right| = \left|\int_{-\infty}^{\infty} \left|u(t)\right|^2 \exp(j2\pi\upsilon t)dt\right|$$

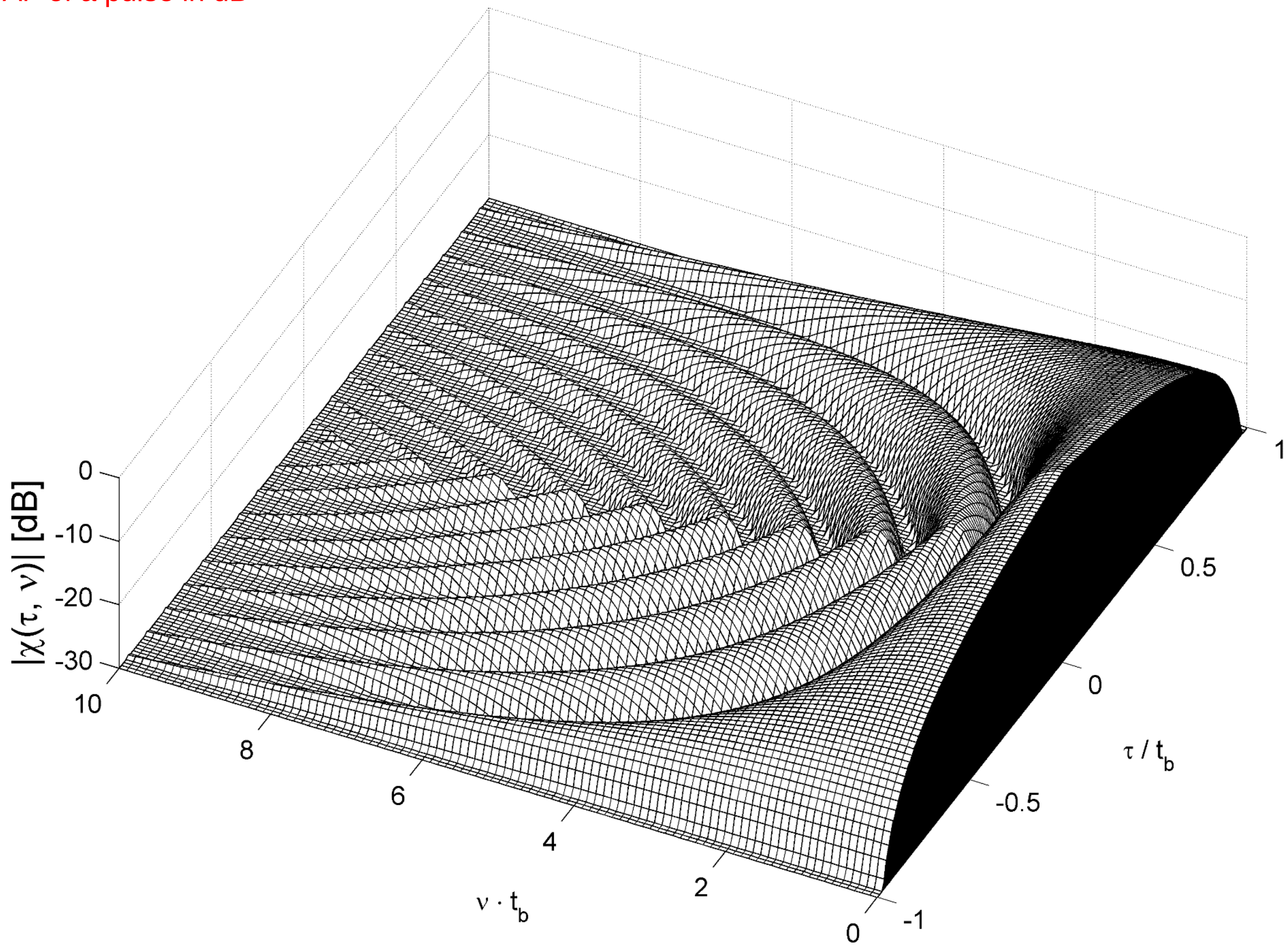
i.e., the cut at zero delay is a function only of the **magnitude** of *u(t)*.





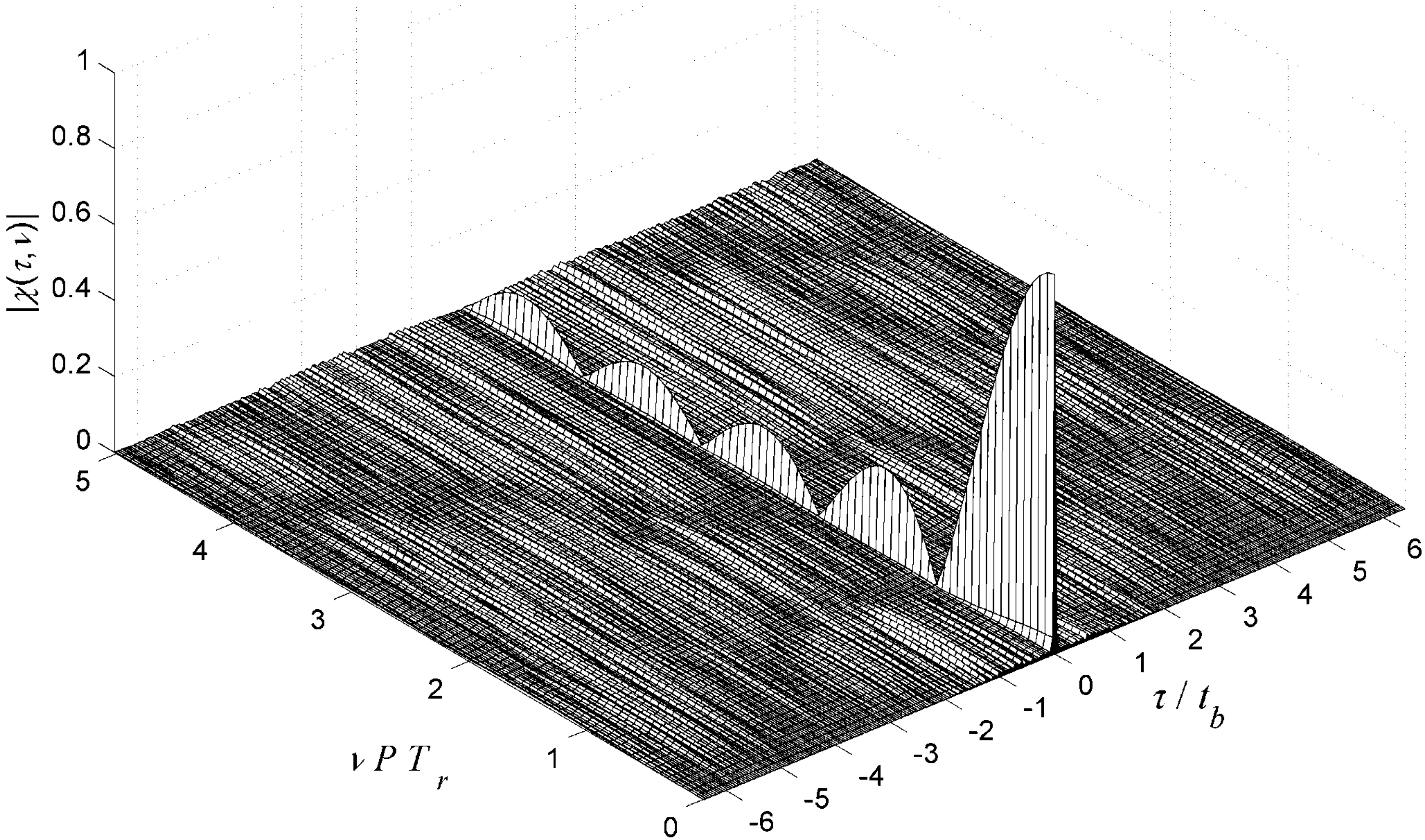


AF of a pulse in dB



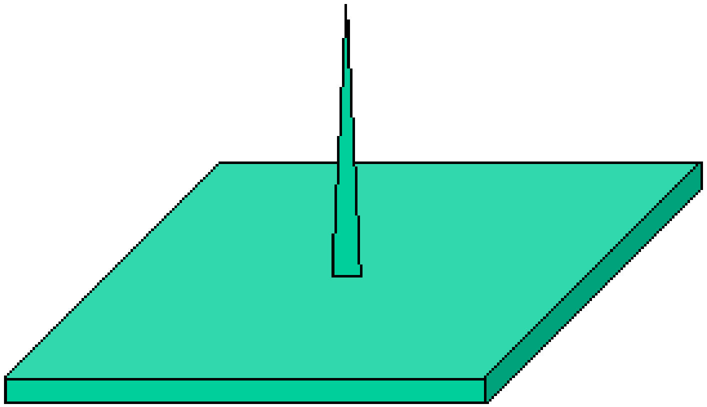
$$\left|\chi(0,\upsilon)\right| = \left| \int_{-\infty}^{\infty} \left|u(t)\right|^2 \exp\left(j\,2\pi\,\upsilon\,t\right)dt \right|$$

Elaborate phase or frequency modulation can lower the sidelobes everywhere except on the Doppler axis.

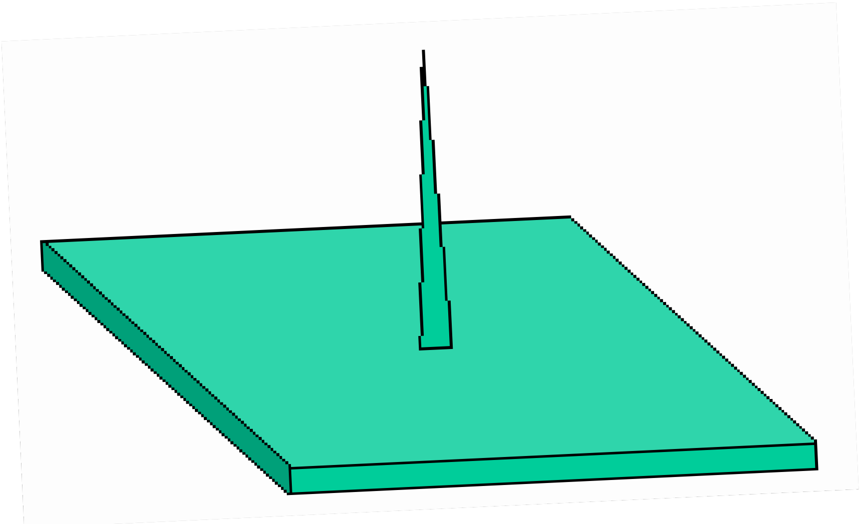


What's an Ideal AF?

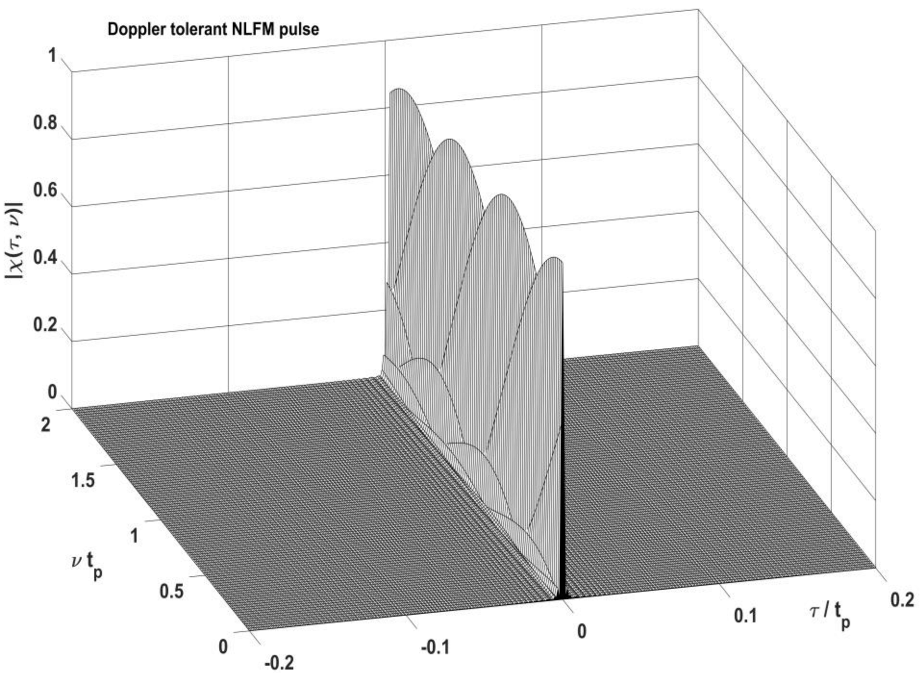
- The "thumbtack" AF is often cited as the ideal
  - no response unless the echo is closely matched to the Doppler for which the filter is designed
  - and a very narrow peak in range yielding good range resolution



Can't get rid of the pedestal because of the "constant volume" property



Thumbtack



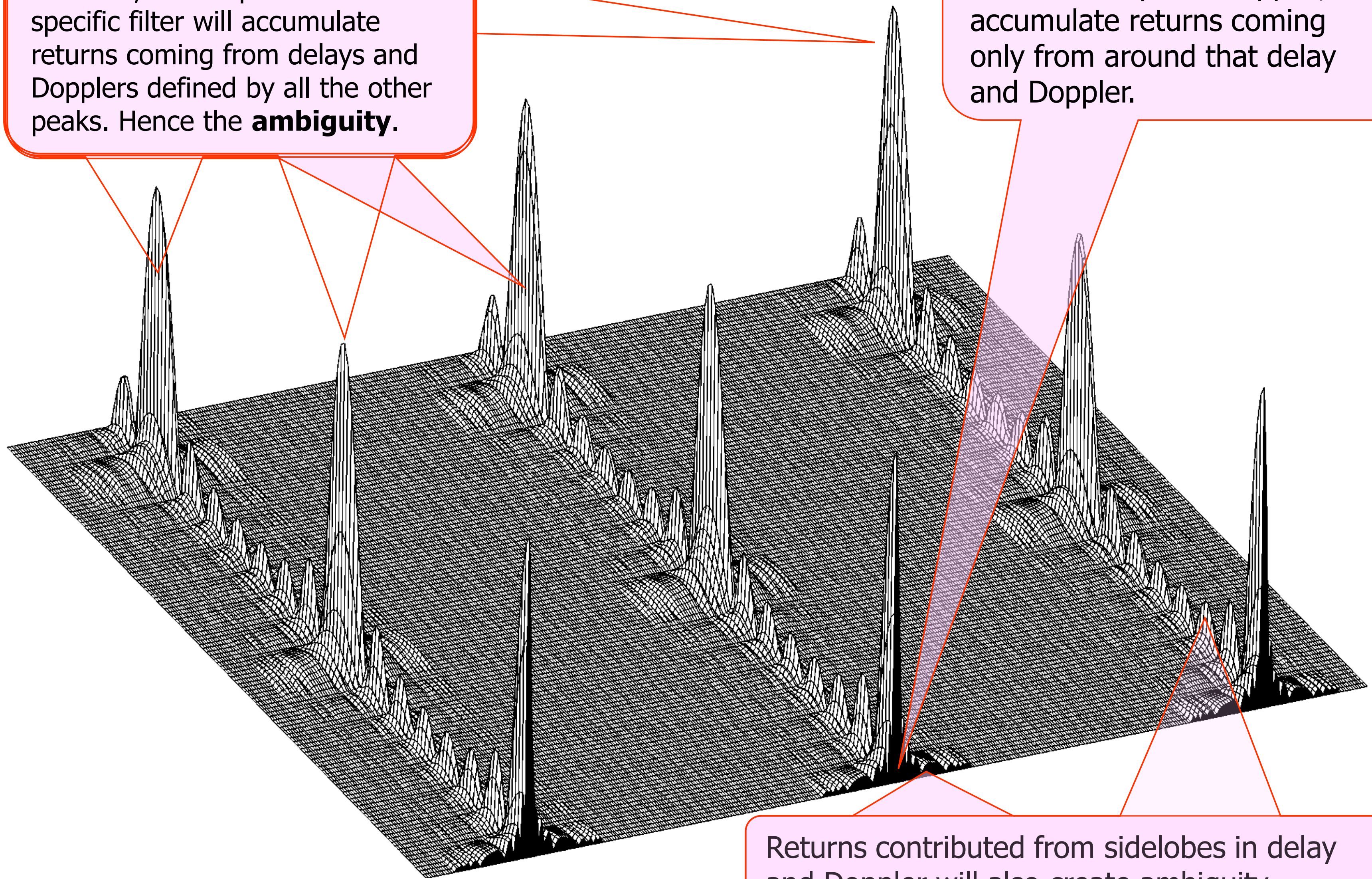
Doppler tolerant ridge

Nadav Levanon, Tel-Aviv University

Why the name "**ambiguity** function"?

However, the output of that specific filter will accumulate returns coming from delays and Dopplers defined by all the other peaks. Hence the **ambiguity**.

We hoped that the output of the filter, designed for the nominal delay and Doppler, will accumulate returns coming only from around that delay and Doppler.



Returns contributed from sidelobes in delay and Doppler will also create ambiguity.

Proof of property #1 - (Peak response)

$$|\chi(\tau,\upsilon)| \le |\chi(0,0)| = 1$$
$$\text{when} \quad E = 1$$

$$|\chi(\tau,\upsilon)| = \left| \int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp(j2\pi\nu t)dt \right|$$

$$|\chi(\tau,\upsilon)|^2 = \left| \int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp(j2\pi\nu t)dt \right|^2$$

Apply Shwarz inequality

$$|\chi(\tau,\upsilon)|^2 \le \int_{-\infty}^{\infty} |u(t)|^2 dt \int_{-\infty}^{\infty} |u^*(t+\tau)\exp(j2\pi\nu t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |u(t)|^2 dt \int_{-\infty}^{\infty} |u^*(t+\tau)|^2 dt = E \cdot E = 1 \cdot 1 = 1$$

$$|\chi(\tau,\upsilon)|^2 \le 1$$

$$\boxed{|\chi(\tau,\upsilon)| \le 1}$$

conjugate of

$$|\chi(\tau,\upsilon)| = 1 \quad \Rightarrow \quad u(t) = u(t+\tau)\exp(-j2\pi\nu t)$$

$$\tau = 0 \;,\; \nu = 0$$

$$\boxed{|\chi(0,0)| = 1}$$

Proof of property #4 - (LFM)

$$u(t) \Leftrightarrow |\chi(\tau,\nu)|$$

$$u(t)\exp\left(j\pi k t^2\right) \Leftrightarrow \left|\chi(\tau,\, \nu - k\tau)\right|$$

$$u_1(t) = u(t)\exp\left(j\pi k t^2\right)$$

$$\chi(\tau,\nu) = \int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp(j2\pi\nu t)dt$$

$$\chi_1(\tau,\nu) = \int_{-\infty}^{\infty} u_1(t)u_1^*(t+\tau)\exp(j2\pi\nu t)dt$$

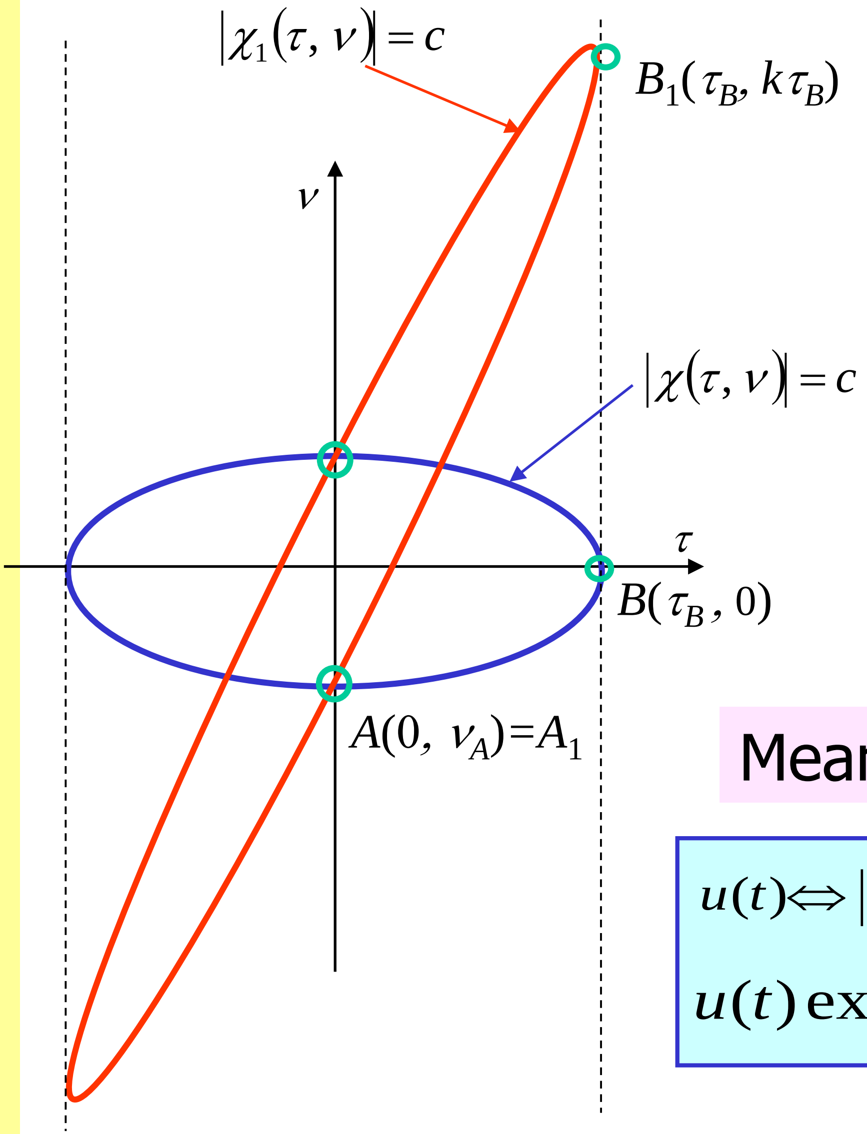
$$\chi_1(\tau,\nu) = \int_{-\infty}^{\infty} u(t)\exp\left(j\pi k t^2\right)u^*(t+\tau)\exp\left[-j\pi k(t+\tau)^2\right]\exp(j2\pi\nu t)dt$$

Rearranging the arguments of the exponents

$$\chi_1(\tau,\nu) = \exp\left(-j\pi k\tau^2\right)\int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp\left[j2\pi(\nu - k\tau)t\right]dt$$

$$|\chi_1(\tau,\nu)| = \left|\int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp\left[j2\pi(\nu - k\tau)t\right]dt\right| = \left|\chi(\tau,\, \nu - k\tau)\right|$$

$$\left|\chi_1\left(\tau_B,\, \underbrace{k\tau_B}_{\nu}\right)\right| = \left|\chi\left(\tau_B,\, \underbrace{k\tau_B}_{\nu} - k\tau_B\right)\right| = \left|\chi\left(\tau_B, 0\right)\right|$$



Meaning of property #4 - (LFM)

$$u(t) \Leftrightarrow |\chi(\tau,\nu)|$$

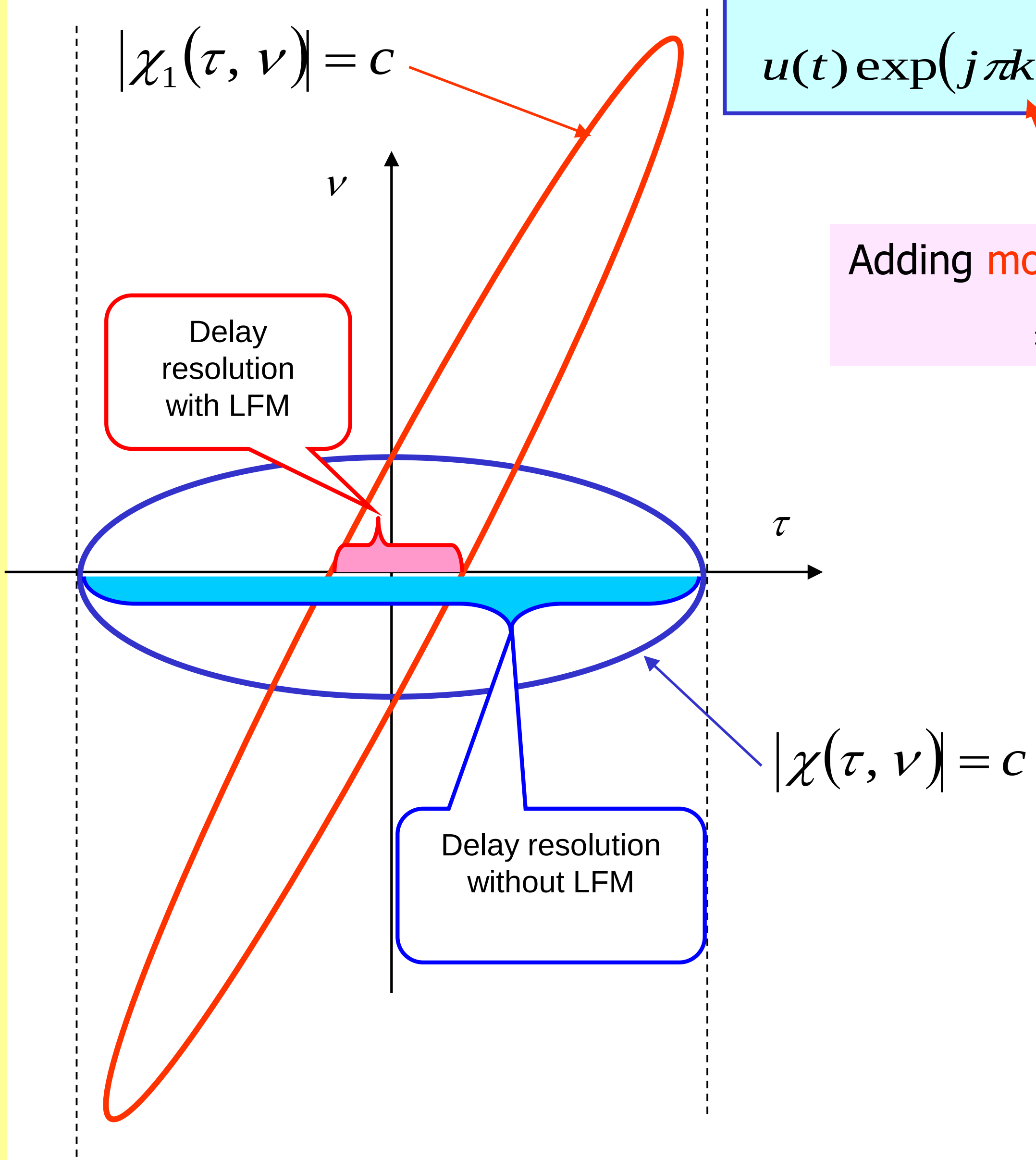
$$u(t)\exp\left(j\pi k t^2\right) \Leftrightarrow |\chi_1(\tau,\, \nu)| = |\chi(\tau,\, \nu - k\tau)|$$

$$u(t) \Leftrightarrow |\chi(\tau, \nu)|$$

$$u(t)\exp\left(j\pi k t^2\right) \Leftrightarrow |\chi_1(\tau, \nu)| = |\chi(\tau, \nu - k\tau)|$$

Adding modulation $\Rightarrow$ increasing bandwidth

$\Rightarrow$ reducing range window

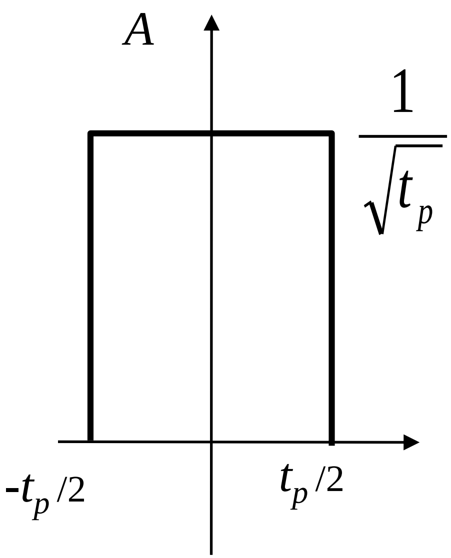


$$\left|\chi(\tau,0)\right| = \left|\int_{-\infty}^{\infty} u(t)u^{*}(t+\tau)dt\right| = \left|R(\tau)\right|$$

Range window $\Leftrightarrow$ Autocorrelation $\Leftrightarrow$ $\mathbf{F}^{-1}$ **{** Power spectrum **}**

AMBIGUITY FUNCTION OF A SINGLE-FREQUENCY PULSE

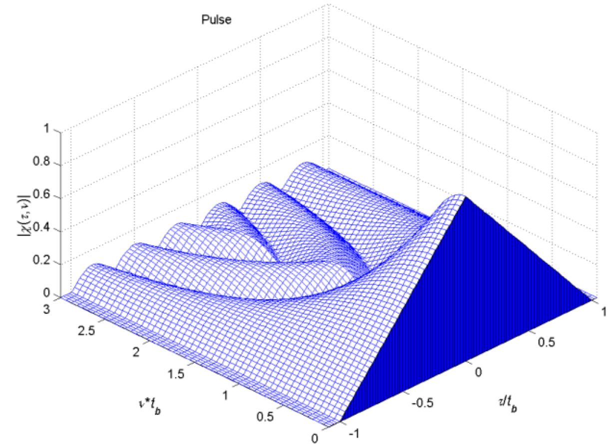
$$|\chi(\tau,\upsilon)| = \left| \int_{-\infty}^{\infty} u(t)u^*(t+\tau)\exp(j2\pi \nu t)dt \right|$$



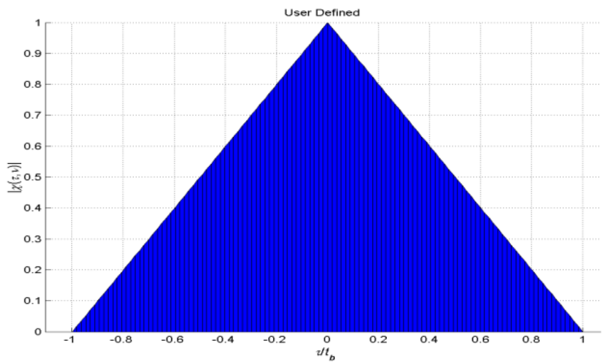
$$\chi(\tau,\nu) = \begin{cases} \dfrac{1}{t_p} \displaystyle\int_{-\frac{t_p}{2}+\tau}^{\frac{t_p}{2}} \exp(j2\pi\nu t)dt & ; \quad 0 < \tau \\ \dfrac{1}{t_p} \displaystyle\int_{-\frac{t_p}{2}}^{\frac{t_p}{2}+\tau} \exp(j2\pi\nu t)dt & ; \quad \tau < 0 \end{cases}$$

$$|\chi(\tau,\upsilon)| = \left| \left(1 - \frac{|\tau|}{t_p}\right) \frac{\sin\left[\pi t_p \nu \left(1 - \frac{|\tau|}{t_p}\right)\right]}{\pi t_p \nu \left(1 - \frac{|\tau|}{t_p}\right)} \right| , \quad |\tau| \le t_p$$

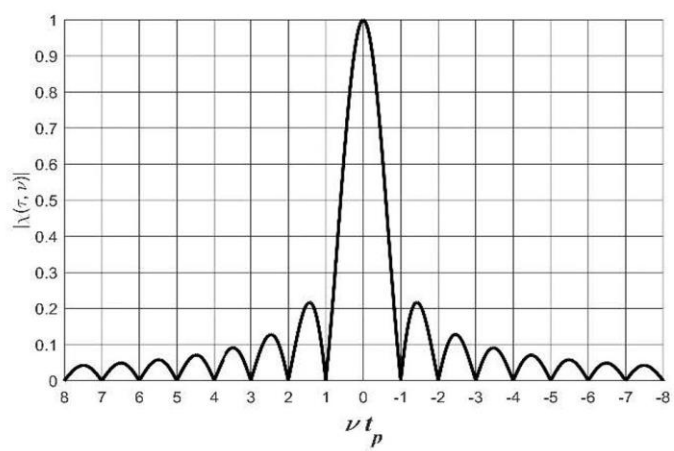
$$\left|\chi(\tau,\upsilon)\right| = \left| \left(1-\frac{|\tau|}{t_p}\right) \frac{\sin\left[\pi\, t_p \nu \left(1-\frac{|\tau|}{t_p}\right)\right]}{\pi\, t_p \nu \left(1-\frac{|\tau|}{t_p}\right)} \right| , \quad |\tau| \le t_p$$



$$\left|\chi(\tau,0)\right| = 1-\frac{|\tau|}{t_p} , \ |\tau| \le t_p$$

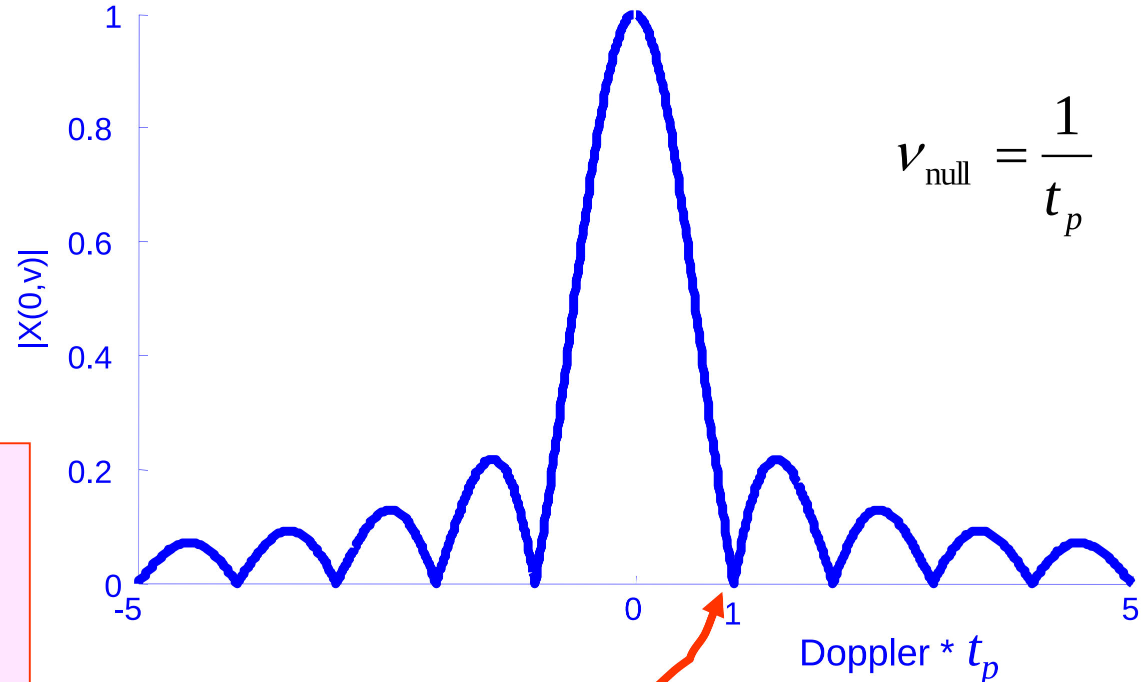


$$\left|\chi(0,\nu)\right| = \left| \frac{\sin\left(\pi\, t_p \nu\right)}{\pi\, t_p \nu} \right|$$



Doppler resolution

$$|\chi(0,\nu)| = \left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right|$$



A signal shifted in Doppler by $1/t_p$ completes exactly one additional cycle during the pulse duration, and produces a value of zero at the output of the matched filter at $\tau = 0$.

<u>Numerical example</u>: $\Delta R = Ct_p/2 = 750\text{m}^*$, $t_p = 5\ \mu\text{s}$, $f_c = 5$ GHz

* With pulse-compression ratio of 100, this will reduce to an acceptable 7.5 m

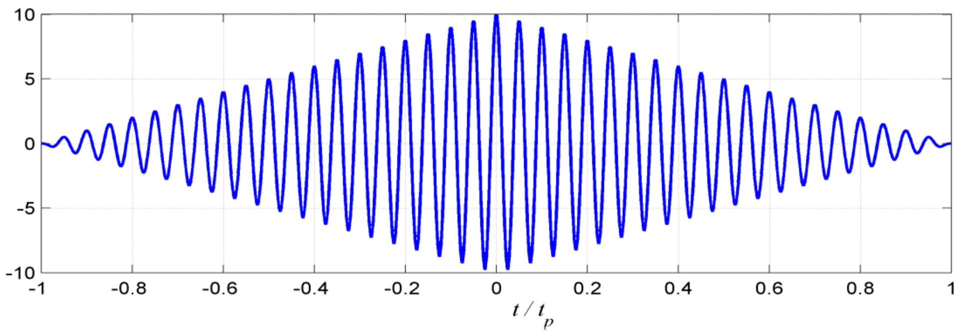
$$V_{\text{null}}\,[\text{m/s}] = \frac{\nu_{\text{null}} C}{2 f_0} = \frac{C}{2 f_0 t_p} = \frac{3\cdot 10^8}{2\cdot 5\cdot 10^9\cdot 5\cdot 10^{-6}} = 6\cdot 10^3\ \text{m/s} = 6\,\text{km/s}$$

This unacceptable velocity resolution will be corrected by using a coherent train of pulses.

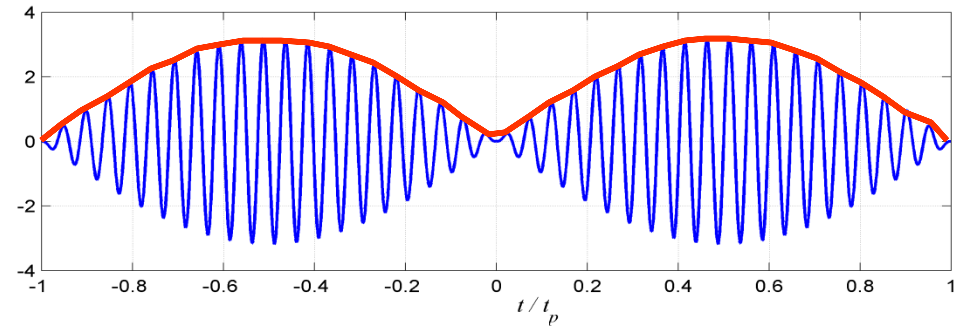
A signal shifted in Doppler by $\mathbf{1/t_p}$ completes exactly one additional cycle during the pulse duration, and produces a value of zero at the output of the matched filter at $\boldsymbol{\tau = 0}$.

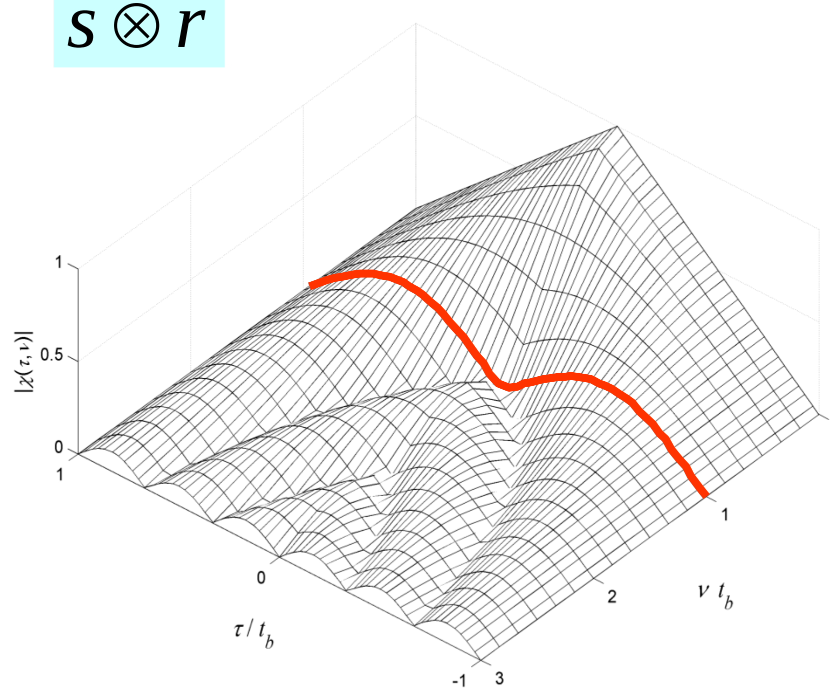
$$s = \sin(2\pi f t),\ 0 \le t \le t_p$$
$$r = \sin\left[2\pi\left(f + \tfrac{1}{t_p}\right)t\right],\ 0 \le t \le t_p$$

$s \otimes s$

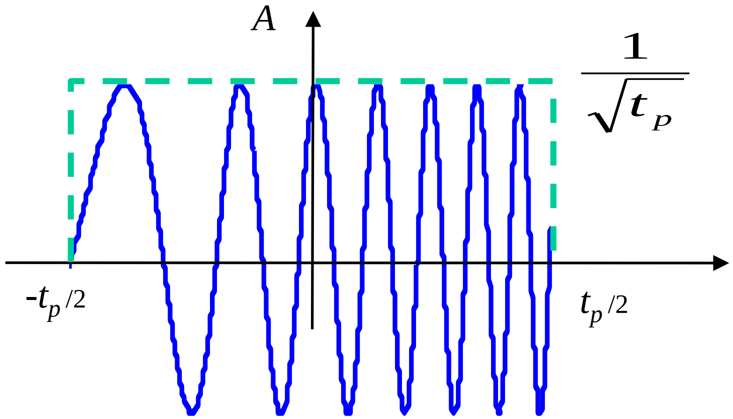


$s \otimes r$





AMBIGUITY FUNCTION OF LINEAR-FM PULSE



$$u(t) = \frac{1}{\sqrt{t_p}} \operatorname{rect}\left(\frac{t}{t_p}\right) \exp\left(j\pi k t^2\right)$$

$$f(t) = \frac{1}{2\pi} \frac{d\left(\pi k t^2\right)}{dt} = kt$$

linear

Instantaneous frequency

Amb. Func. of single-frequency pulse

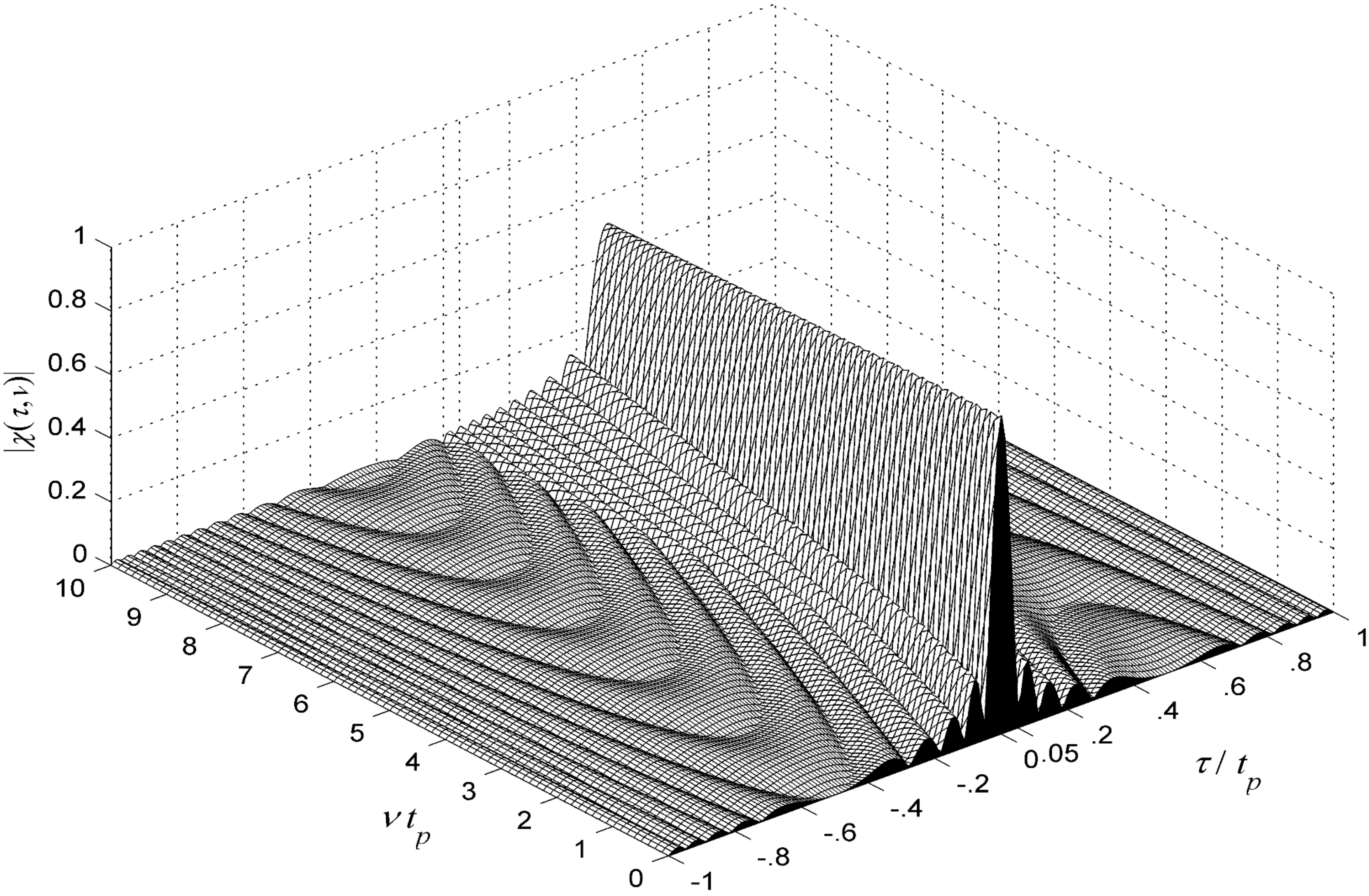
$\nu - k\tau$

$$|\chi(\tau,\nu)| = \left| \left(1 - \frac{|\tau|}{t_p}\right) \frac{\sin\left[\pi t_p \nu \left(1 - \frac{|\tau|}{t_p}\right)\right]}{\pi t_p \nu \left(1 - \frac{|\tau|}{t_p}\right)} \right| , \quad |\tau| \le t_p$$

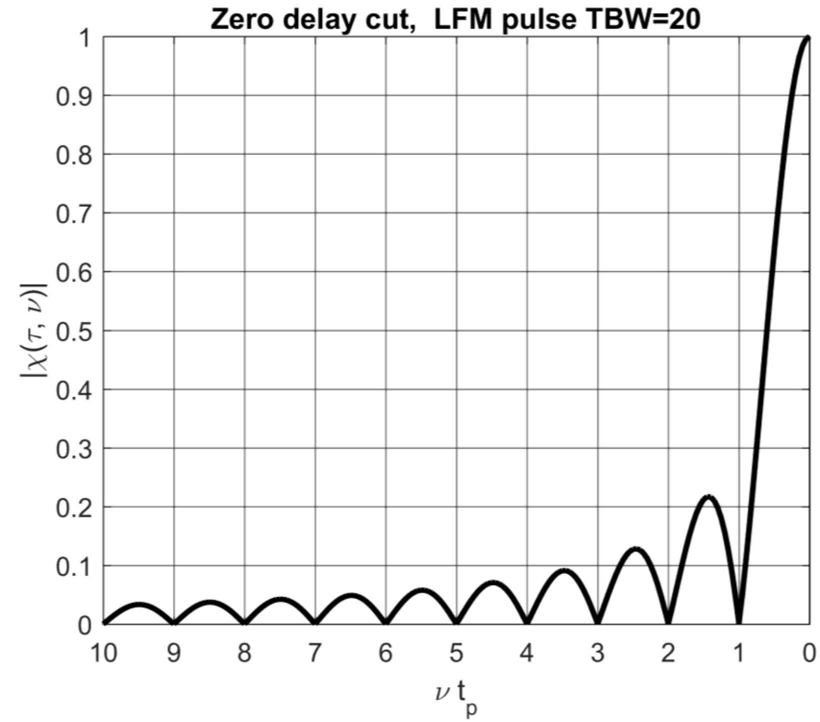
$\nu - k\tau$

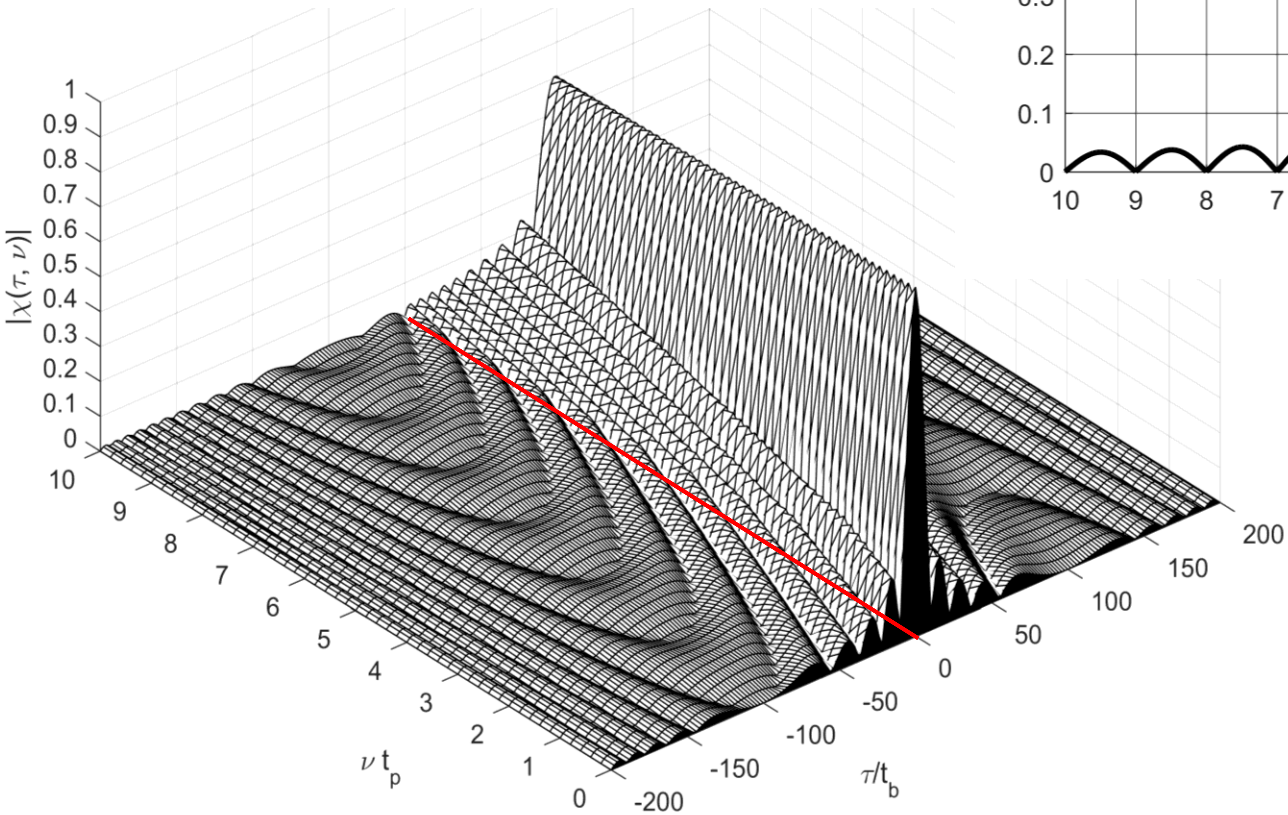
$$|\chi(\tau,\nu)| = \left| \left(1 - \frac{|\tau|}{t_p}\right) \frac{\sin\left[\pi t_p (\nu - k\tau) \left(1 - \frac{|\tau|}{t_p}\right)\right]}{\pi t_p (\nu - k\tau) \left(1 - \frac{|\tau|}{t_p}\right)} \right| , \quad |\tau| \le t_p$$

Ambiguity function of an LFM pulse with TBW = 20

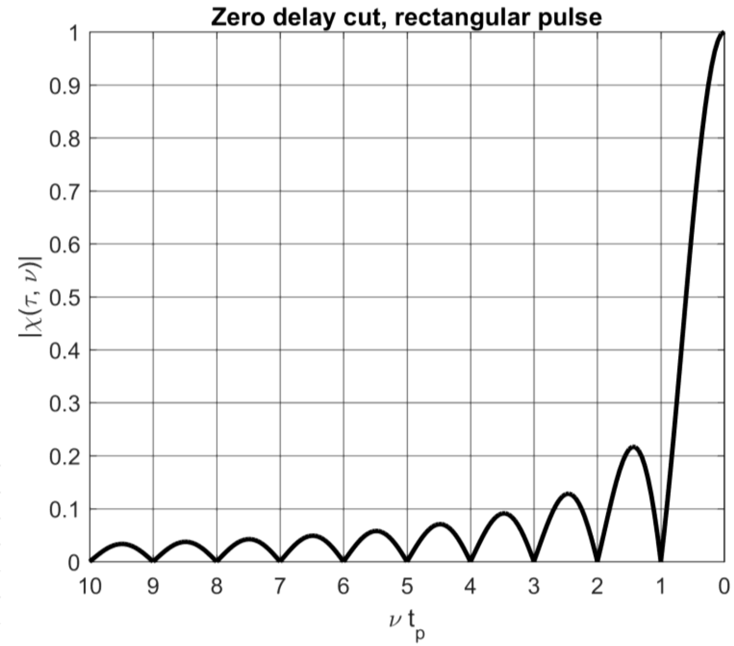


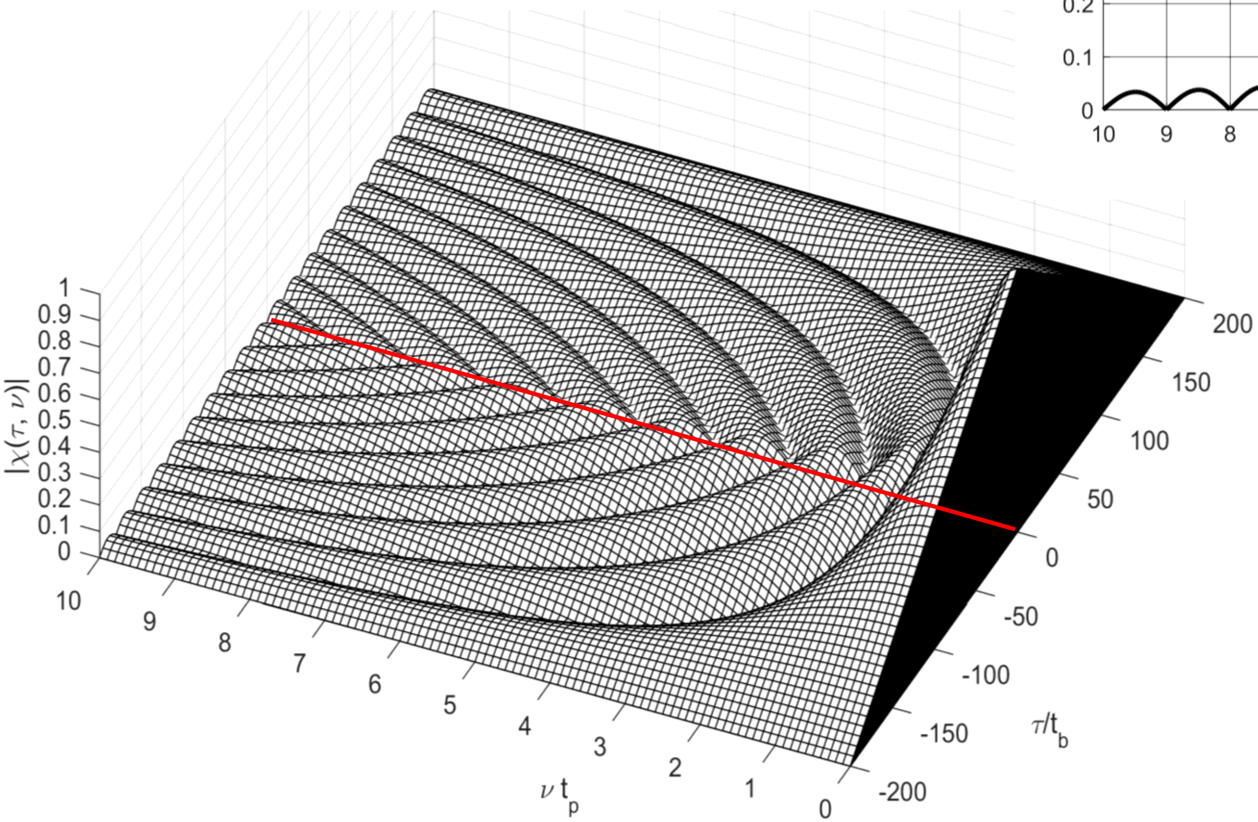
$$\left|\chi(0,\nu)\right| = \left|\frac{\sin(\pi t_p \nu)}{\pi t_p \nu}\right|$$

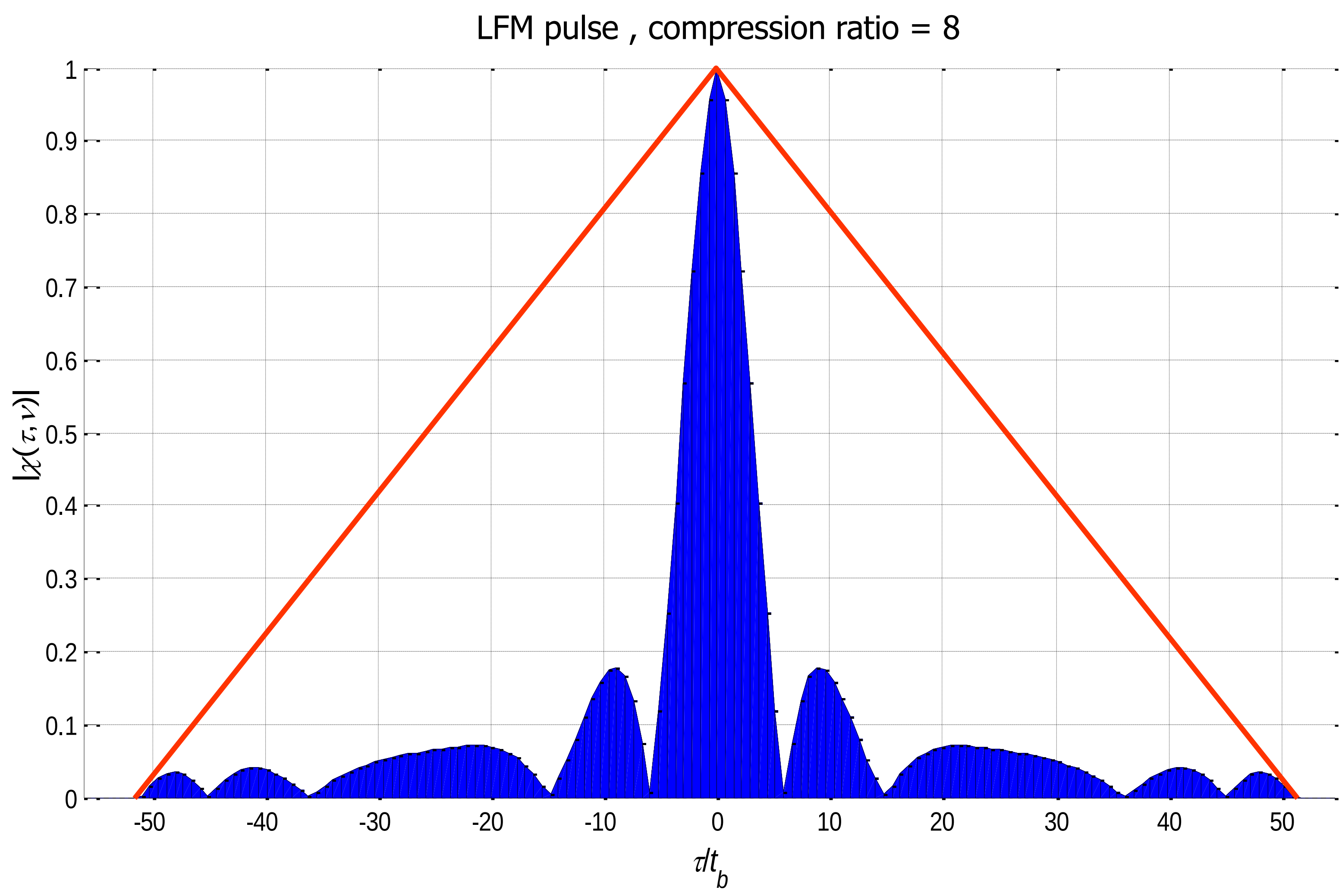


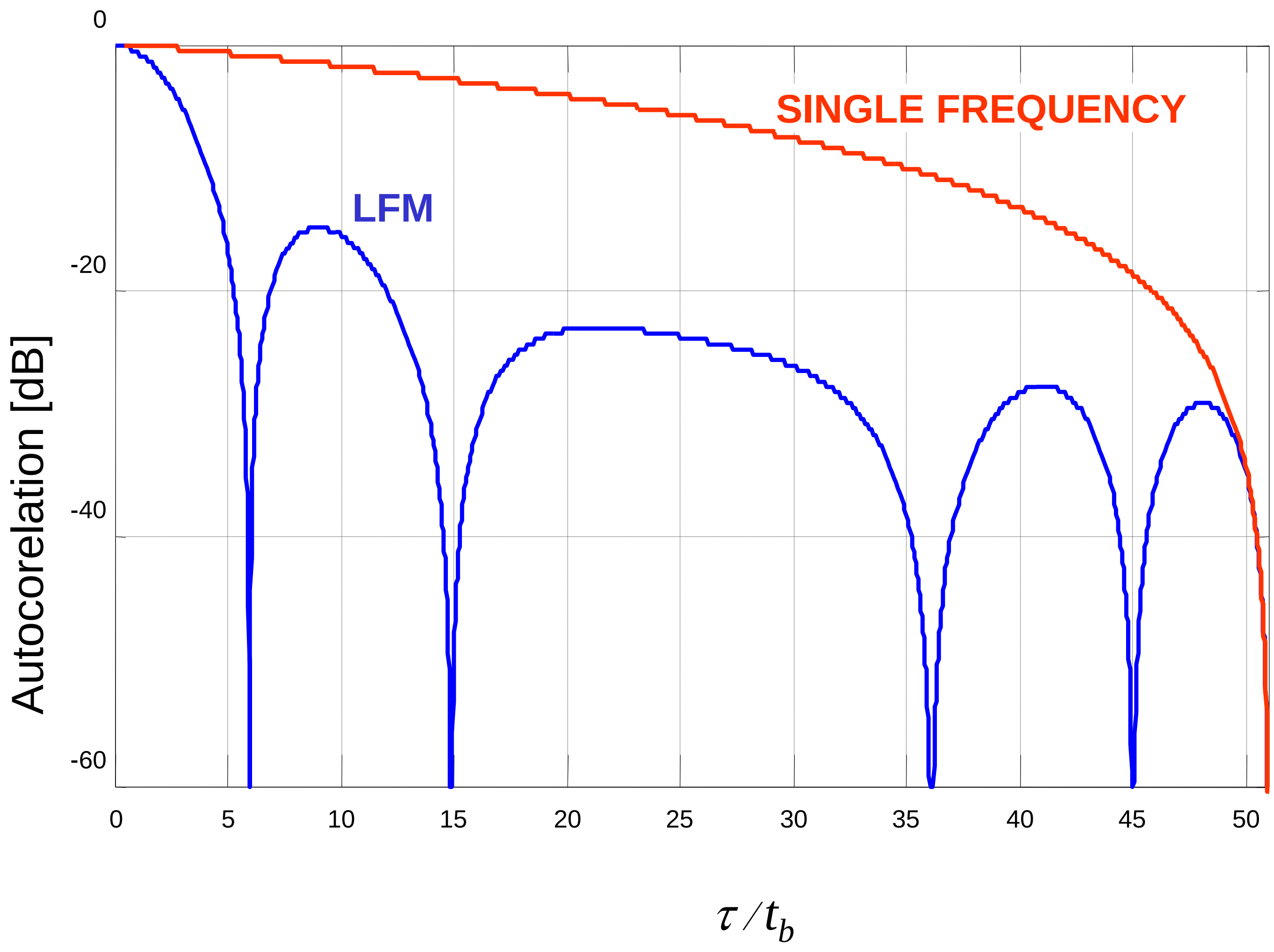


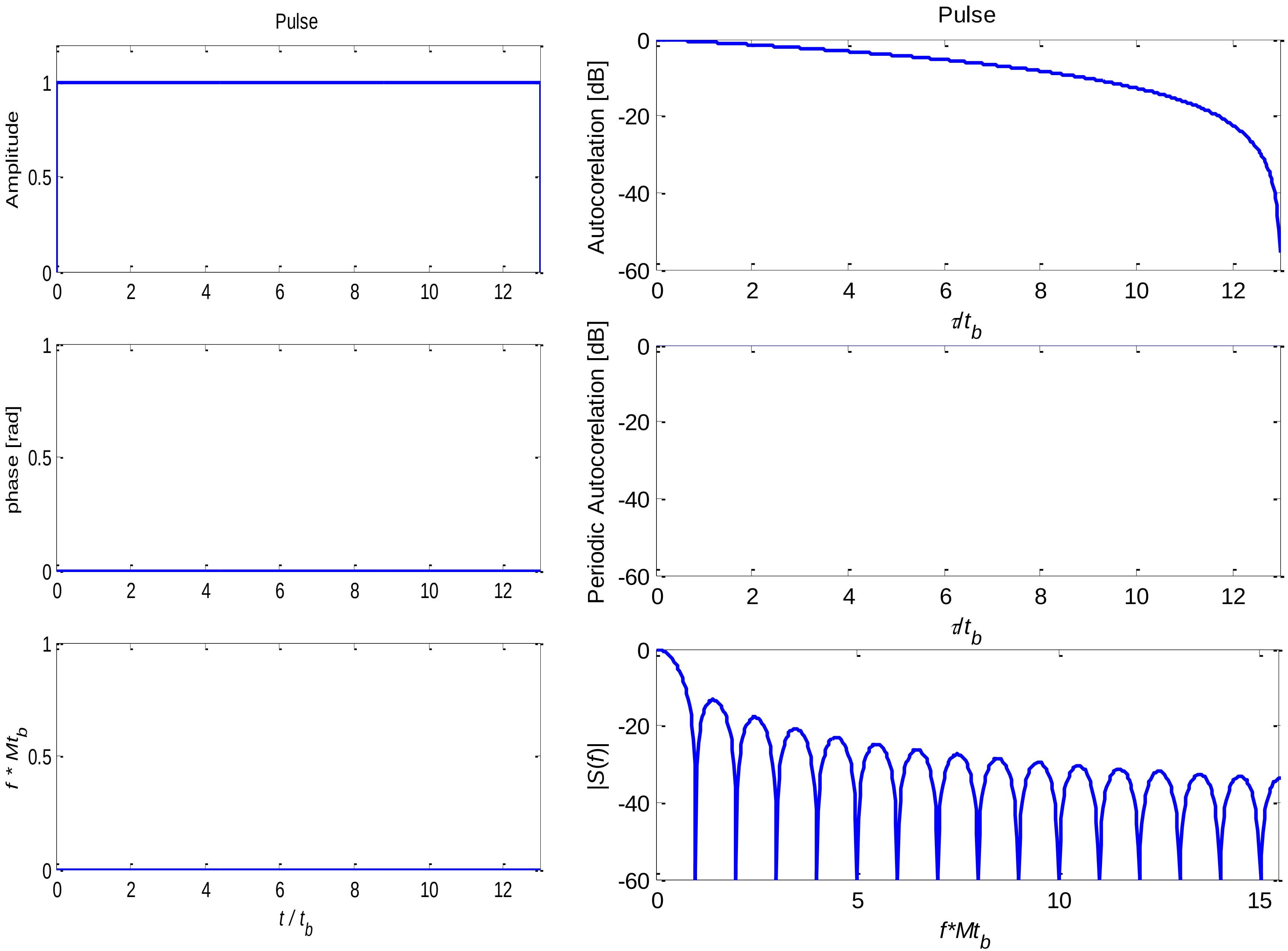
$$|\chi(0,\nu)| = \left| \frac{\sin(\pi t_p \nu)}{\pi t_p \nu} \right|$$

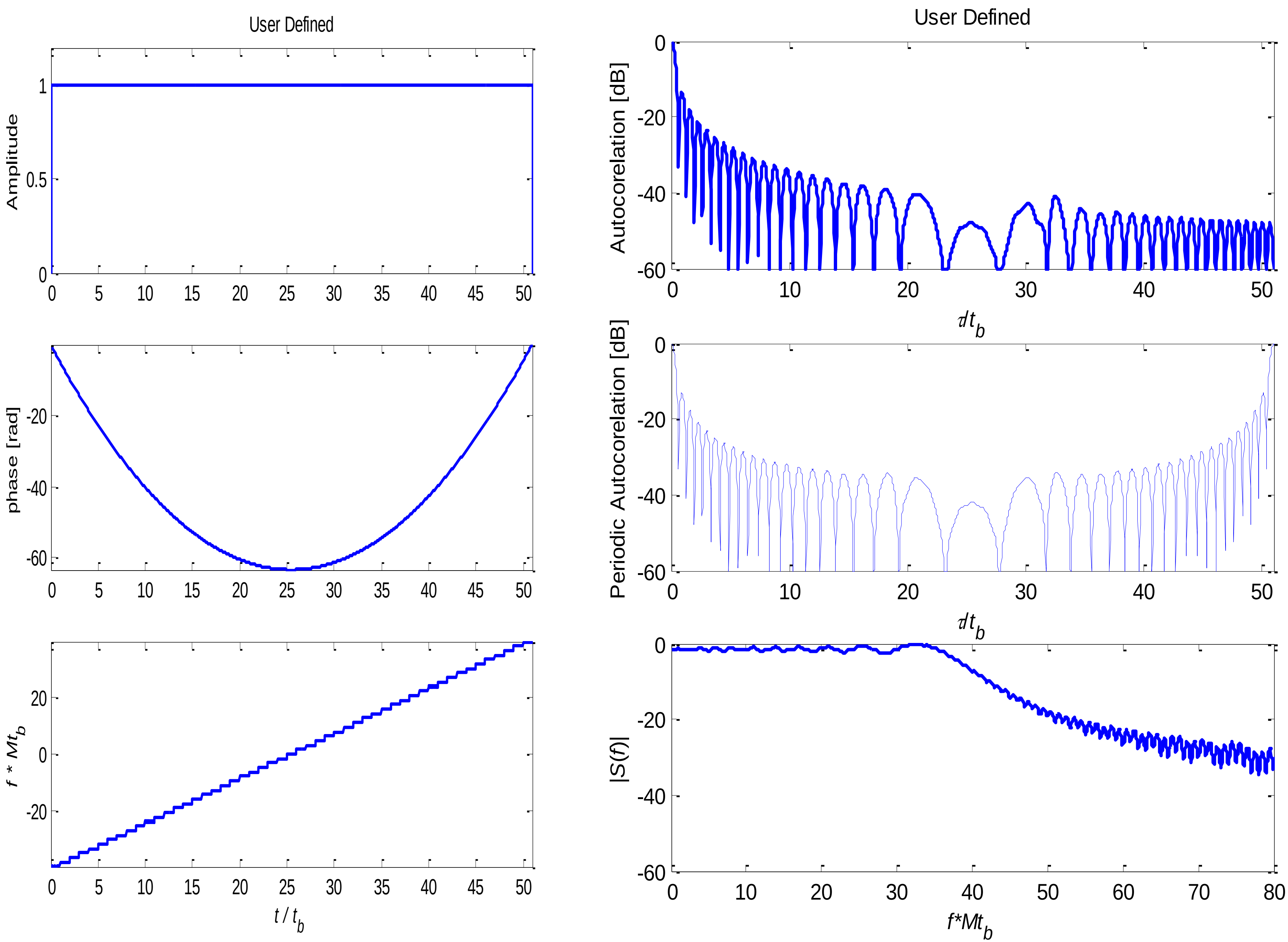





LFM pulse , compression ratio = 8
$|\chi(\tau,\nu)|$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-50
-40
-30
-20
-10
0
10
20
30
40
50
τ/t_b

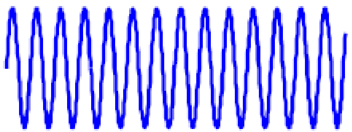
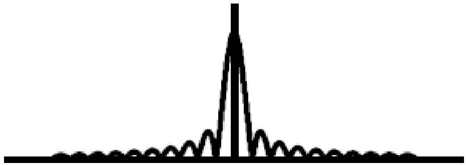
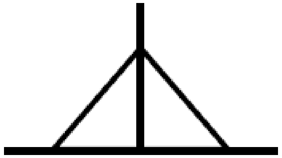

SINGLE FREQUENCY
LFM
Autocorelation [dB]
0
-20
-40
-60
0
5
10
15
20
25
30
35
40
45
50
τ / t_b

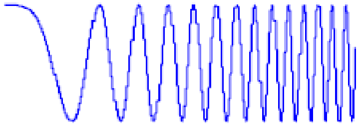
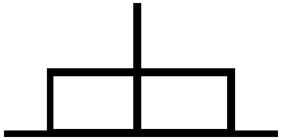
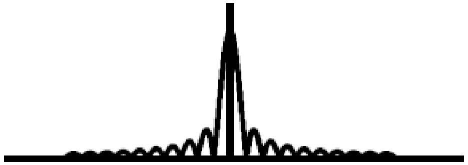

Pulse
Amplitude
phase [rad]
f * Mt_b
t / t_b
Pulse
Autocorelation [dB]
τ/t_b
Periodic Autocorelation [dB]
τ/t_b
|S(f)|
f*Mt_b


User Defined
Amplitude
phase [rad]
f * Mt_b
t / t_b
User Defined
Autocorelation [dB]
τ/t_b
Periodic Autocorelation [dB]
τ/t_b
|S(f)|
f*Mt_b

PULSE COMPRESSION

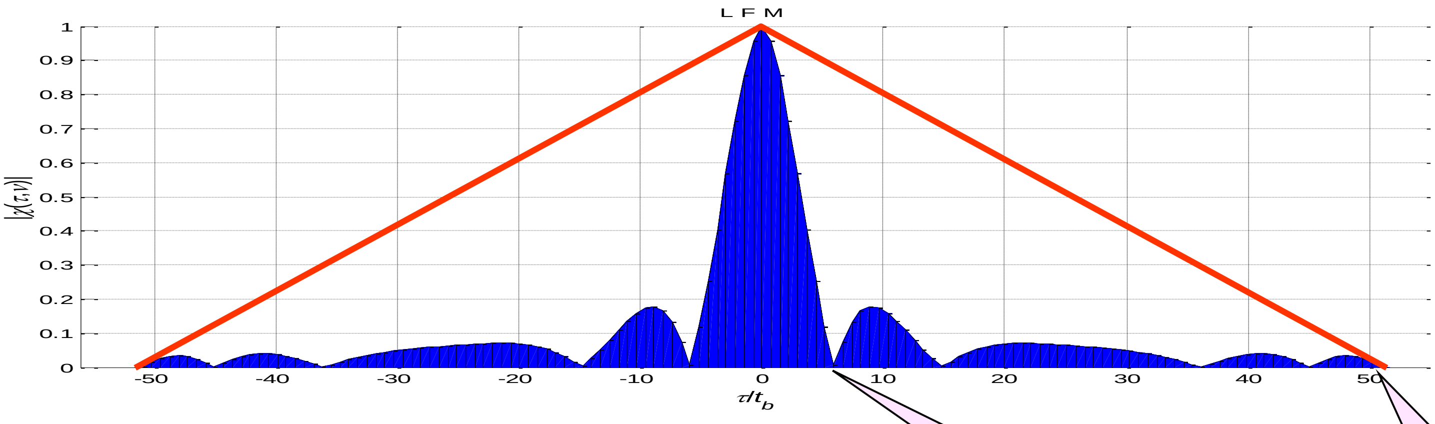
LFM *vs.* Simple Pulse

Time Domain	Frequency Domain	Matched Filter Output
$x(t)$	$\lvert X(\omega)\rvert^2$	$y(t)$
		
Simple Pulse	Sinc Squared	Triangular Response

Time Domain	Frequency Domain	Matched Filter Output
$x(t)$	$\lvert X(\omega)\rvert^2$	$y(t)$
		
LFM Waveform	Approximate Rectangular Spectrum	Sinc-like Response

Nadav Levanon, Tel-Aviv University

COMPRESSION RATIO



$$\pi t_p k \tau \left(1-\frac{|\tau|}{t_p}\right) = \pi \quad \Rightarrow \quad \text{first null}$$

$$|\chi(\tau,0)| = \left| \left(1-\frac{|\tau|}{t_p}\right) \frac{\sin\left[\underbrace{\pi t_p k \tau \left(1-\frac{|\tau|}{t_p}\right)}\right]}{\pi t_p k \tau \left(1-\frac{|\tau|}{t_p}\right)} \right| , \quad |\tau| \le t_p$$

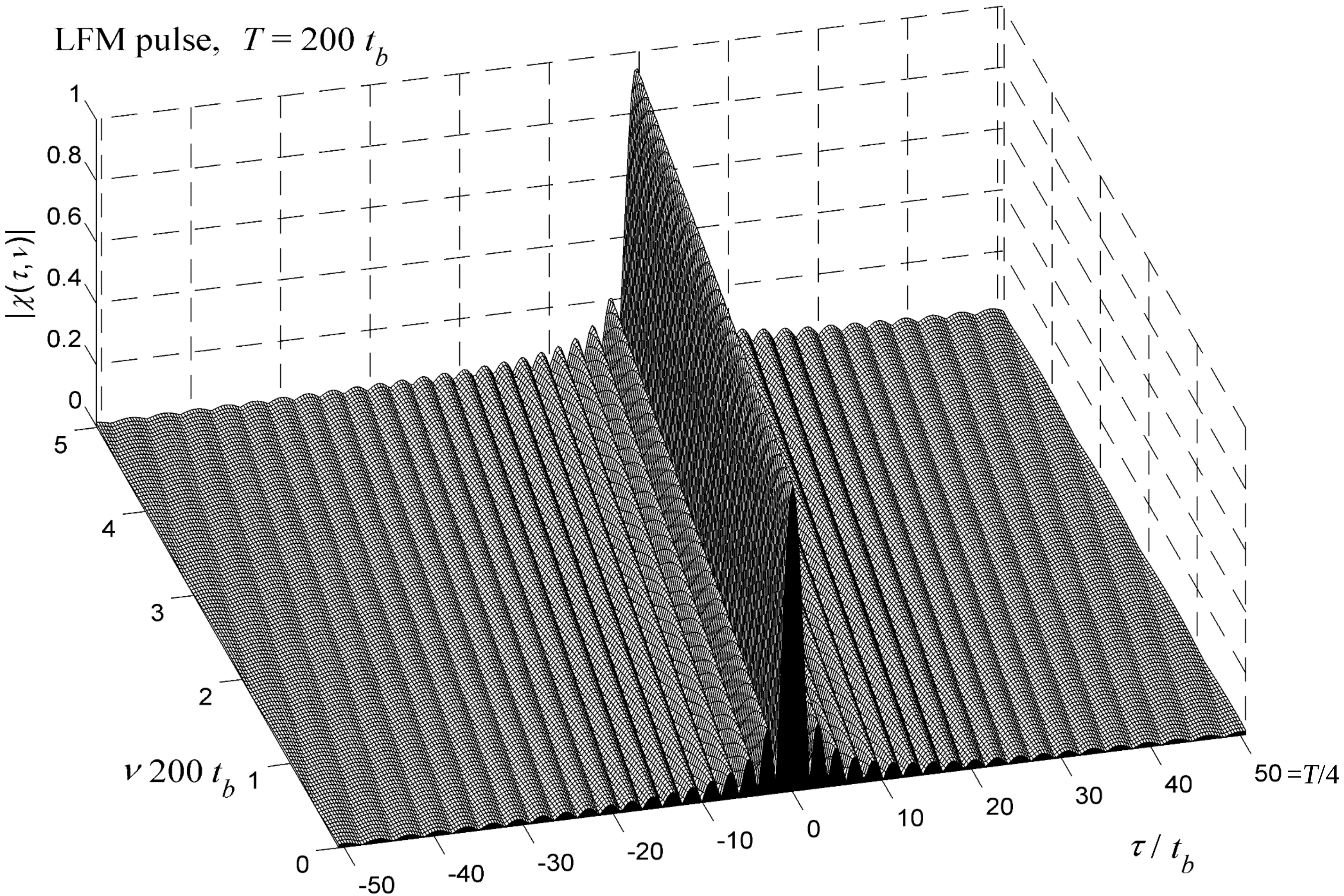
$$\tau_{\text{1'st null}} = \frac{t_p}{2} - \sqrt{\frac{t_p^2}{4} - \frac{1}{k}} \underset{kt_p^2 >> 4}{\approx} \frac{1}{kt_p}$$

$kt_p = \Delta f$ is the total frequency deviation during the pulse

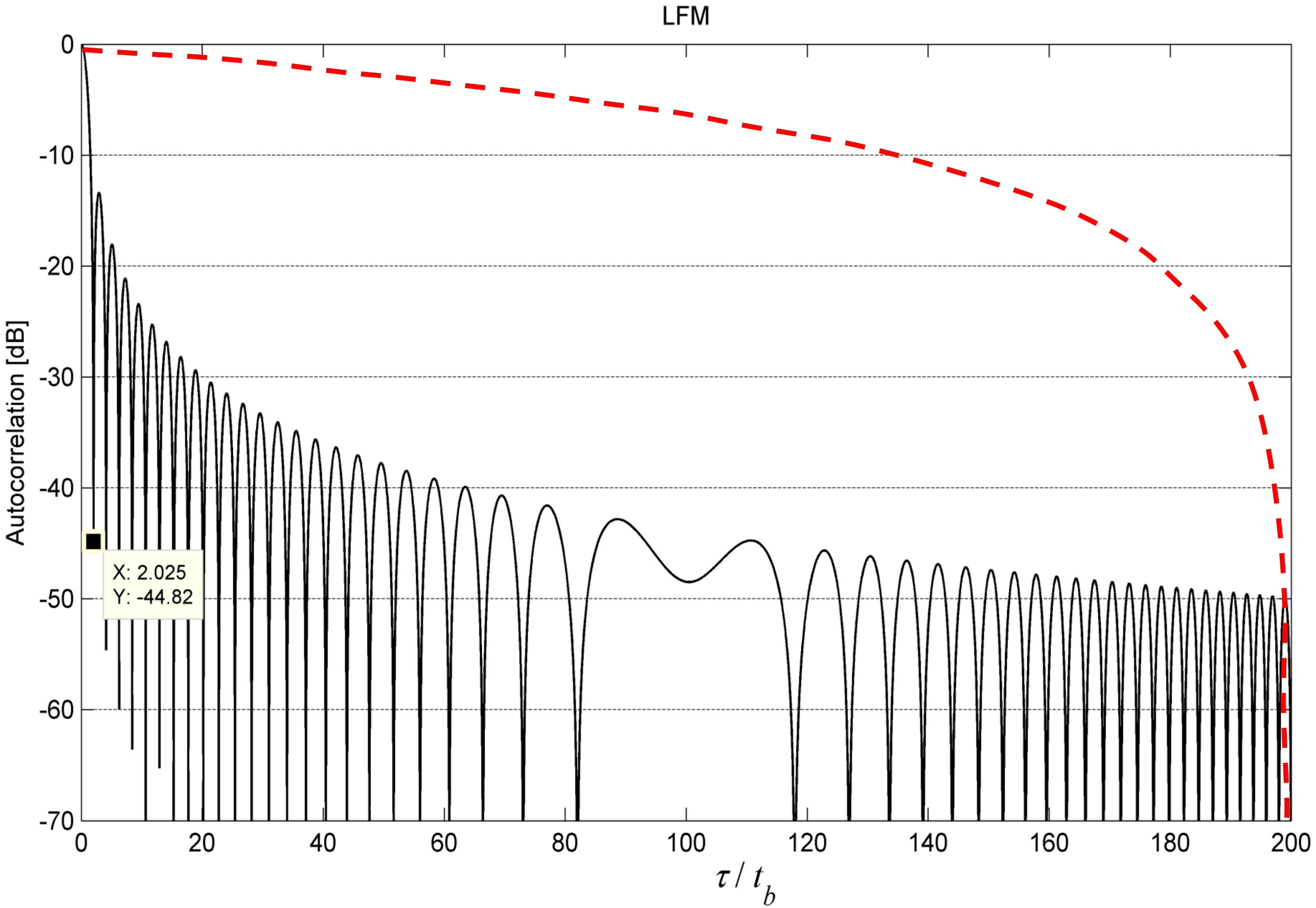
$$\tau_{\text{1'st null}} \underset{\Delta f\, t_p >> 4}{\approx} \frac{1}{\Delta f} \quad , \qquad \text{Compression ratio} = D = \frac{t_p}{\tau_{\text{1'st null}}} = \Delta f \, t_p$$

= Time* Bandwidth (TBW)

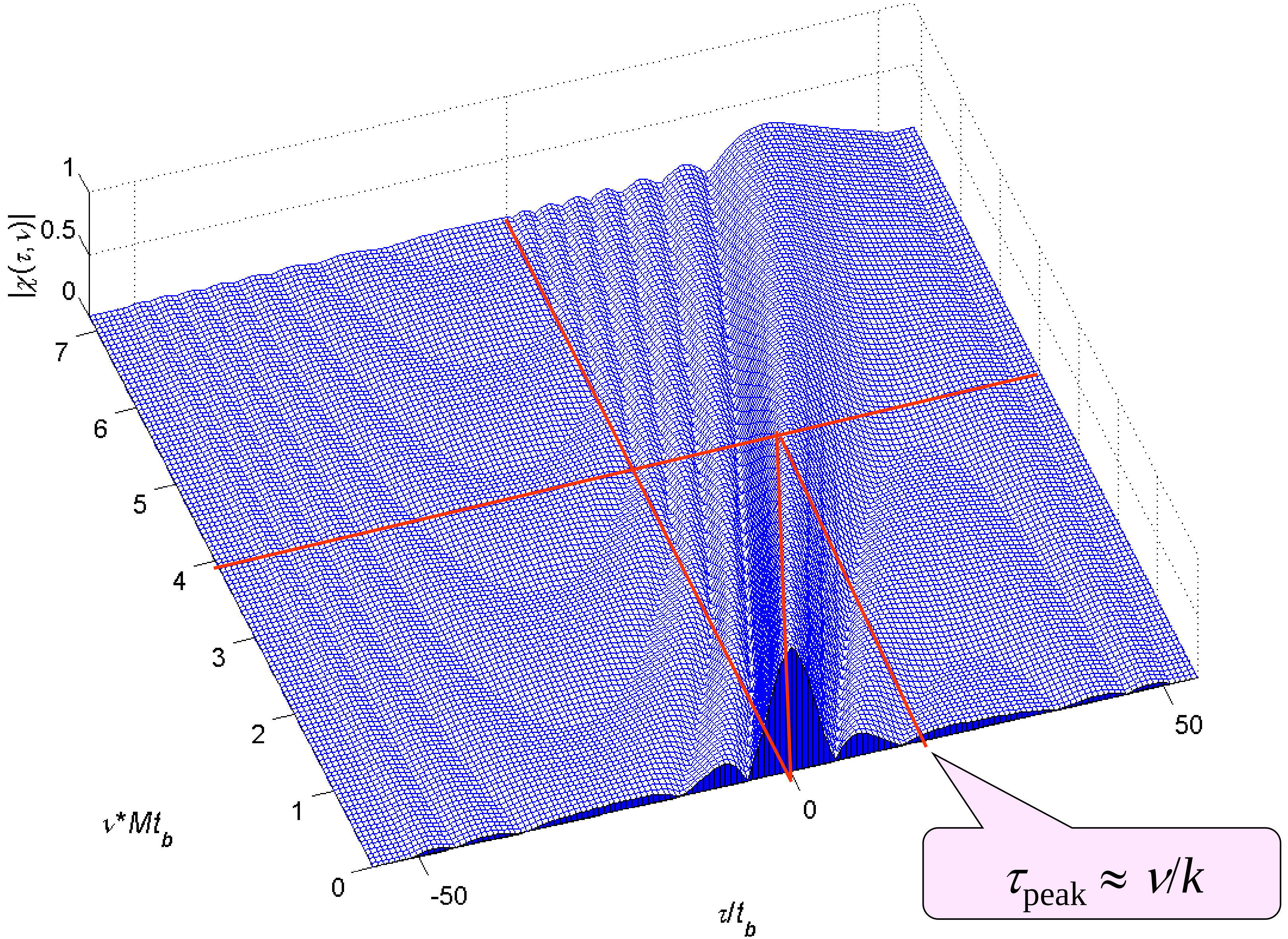
LFM pulse with compression ratio = 100



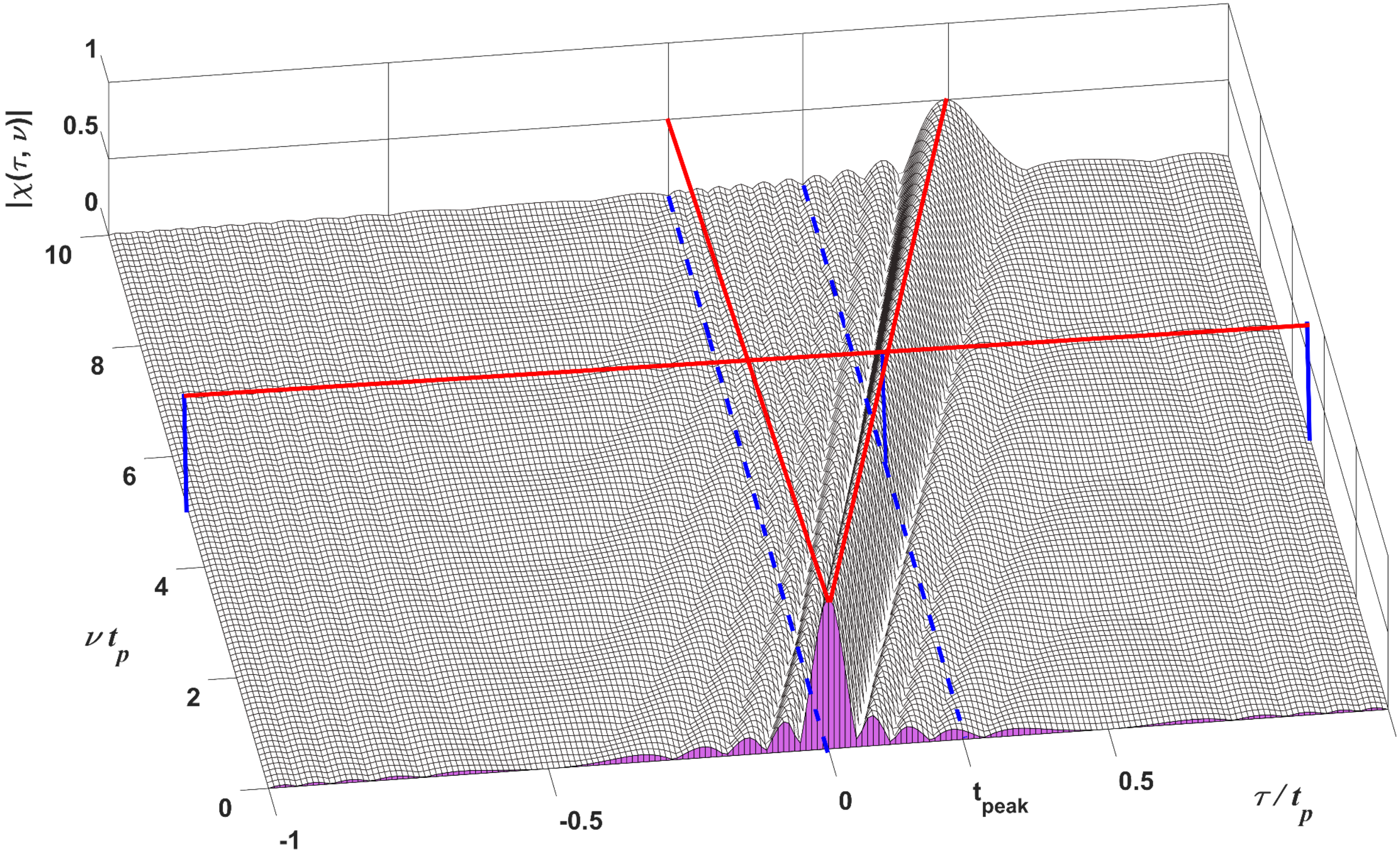
LFM pulse with compression ratio = 100



Delay-Doppler coupling in LFM



Delay-Doppler coupling in LFM



$$t_{\text{peak}} \approx \nu / k$$

The ambiguity function describes the response of an ideal matched filter (implemented in the receiver) to the transmitted signal. Usually this is the best possible response (at least from the SNR viewpoint).

In some cases, usually for simplicity, the receiver implements another processor. E.g., in LFM signal, stretch processor is sometimes used.

The performances of an unmatched filter are usually lower, and their evaluation more difficult.

The AF resembles the Cramer-Rao lower bound in parameter estimation. It is simpler to calculate than the actual performances of a non-ideal estimator, and it serves as a lower bound on the attainable accuracy.

An LFM pulse provides good delay resolution but no Doppler resolution.

The intensity of the output of the matched filter is almost the same for any Doppler shift.

Good Doppler resolution can be obtained from a **coherent train of pulses.**