Correspondence

Passive Array Tracking of a Continuous Wave Transmitting Projectile

Abstract

Differential Doppler measurements by a passive array are used to track an unstable continuous wave (CW) source moving in a ballistic trajectory, e.g., a projectile carrying a proximity fuse. The ballistic equations of motion couple frequency measurements at various sections along the track with the track parameters at any arbitrary time, e.g., at impact. A nonlinear weighted least-square method is used to estimate the track parameters, and the resulting error covariance matrix is derived. A numerical example demonstrates the relative contributions of various frequency measurements to the estimation accuracy.

Introduction

Tracking artillery and mortar projectiles is currently done by radars. The increase use of proximity fuses (PF) converts the projectile from a passive to an active target, making passive tracking feasible. A continuous wave (CW) transmitting source is the most difficult to track. Lack of modulation means no time markers, preventing measurement of the delay parameter. However, because many PF are of the CW type, this paper does not utilize delay, but only Doppler frequency measurements. The PF transmitter is usually a fundamental power oscillator whose frequency may change slowly during the flight. For this reason only differential Doppler shift is used.

Fig. 1 shows the relevant geometrical picture. The source (projectile) follows the indicated track and radiates a sinusoidal signal towards a spatially separated array of receivers. Each receiver is contaminated by Gaussian noise so that the actual waveshape observed at the ith receiver is given by

$$P_i(t) = A_i \sin(w_i t - \theta_i) + n_i(t) . \tag{1}$$

All the information concerning source trajectory is contained in the relative frequency and phase of the signal to several receivers; however, the only quantity accessible to measurement is the differential frequency shift $(w_i - w_j)$ between pairs of receivers. We assume perfect propagation conditions in the medium between source and receiver. In this case, the instantaneous differential Doppler shift is given by

$$w_i - w_j = -(\dot{r}_i - \dot{r}_j)\, 2\pi/\lambda \tag{2}$$

where λ is the radiated wavelength; r_i and $\dot{r}_i$ are, respectively, range and range rate between source and ith re-

Manuscript received June 8, 1979; revised February 2, 1980.

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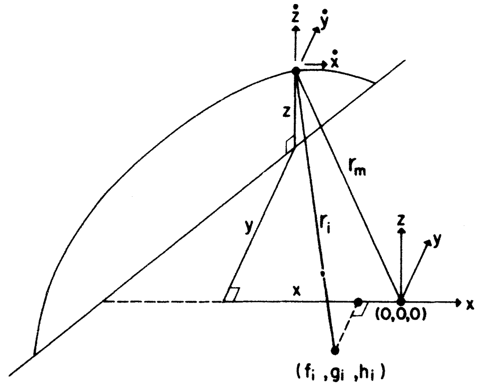

Fig. 1. Array-source geometry.

ceiver. In a Cartesian coordinate system r_i is given by

$$r_i = [(x - f_i)^2 + (y - g_i)^2 + (z - h_i)^2]^{1/2} \tag{3}$$

where x, y, z are the instantaneous position components measured in some convenient coordinate system and (f_i, g_i, h_i) determines the location of the ith receiver in that coordinate system. Differentiating (3) with respect to t yields

$$\dot{r}_i = [(x - f_i)\dot{x} + (y - g_i)\dot{y} + (z - h_i)\dot{z}] / \{[(x - f_i)^2 + (y - g_i)^2 + (z - h_i)^2]^{1/2}\} . \tag{4}$$

Until now no assumption concerning source trajectory has been made. We shall assume that the ballistic motion of the projectile is governed by two accelerations: gravity and drag. Simplified expressions for the accelerations along the three directions are

$$\ddot{x} = - \mathrm{B}e^{-Lz} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \dot{x} \tag{5a}$$

$$\ddot{y} = - \mathrm{B}e^{-Lz} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \dot{y} \tag{5b}$$

$$\ddot{z} = - \mathrm{B}e^{-Lz} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2} \dot{z} - g \tag{5c}$$

where g is the acceleration due to gravity (= 9.8m/s^2) and L represents the change of air density with altitude (= 10^{-4}m^{-1}). B, the ballistic constant, is given by the formula

$$B = \eta_0 C_D S/2M \tag{6}$$

where η_0 is the air density at the array altitude ($z = 0$). C_D is the drag coefficient. S and M are, respectively, the effective area and the mass of the projectile. From (5a) and (5b) one can clearly see that $\ddot{x}/\ddot{y} = \dot{x}/\dot{y}$. This presentation, therefore, assumes that the projectile maintains the direction of its horizontal component of flight.

If all the constants appearing in (5) are known, the parameter vector

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$$\mathrm{s}^{(n)} = (x_n, y_n, z_n, \dot{x}_n, \dot{y}_n, \dot{z}_n)^{\mathrm{T}} \tag{7}$$

of position and velocity components at some arbitrary time instant $t = t_n$ along the path completely characterizes source trajectory (as a function of time). For our purposes t_n may be associated with the time of impact. The set of track parameters at a different time instant can be obtained from the components of $\mathbf{s}^{(n)}$ by backward (forward) integration of the equations of motion [(5a) through (5c)] along the time elapsed using the interpolation formulas

$$\mathbf{x}_{k-1} = \mathbf{x}_k - \Delta t \, \dot{\mathbf{x}}_k \tag{8a}$$

$$\dot{\mathbf{x}}_{k-1} = \dot{\mathbf{x}}_k - \Delta t \, \ddot{\mathbf{x}}_k . \tag{8b}$$

The vectors in (8) are

$$\mathbf{x}_k = (x_k, y_k, z_k)^{\mathrm{T}} \tag{9a}$$

$$\dot{\mathbf{x}}_k = (\dot{x}_k, \dot{y}_k, \dot{z}_k)^{\mathrm{T}} \tag{9b}$$

$$\ddot{\mathbf{x}}_k = (\ddot{x}_k, \ddot{y}_k, \ddot{z}_k)^{\mathrm{T}} \tag{9c}$$

where $\Delta t = t_k - t_{k-1}$. A set of higher order interpolation formulas is given later.

Our objective is, therefore, to estimate the set of track parameters $\mathrm{s}^{(n)}$ using differential Doppler shift measurements.

Estimation of Track Parameters

If the array consists of m receivers, there are $(m - 1)$ linearly independent differential Doppler shifts during each observation period, obtainable by choosing one as a reference receiver and comparing its input with that of every other receiver. Thus the total set of differential Dopplers consists of

$$\Delta \mathbf{w} = (\Delta w_1^{(1)}, \Delta w_2^{(1)}, \ldots, \Delta w_{m-1}^{(1)}, \Delta w_1^{(2)}, \ldots, \Delta w_{m-1}^{(2)}, \ldots)^{\mathrm{T}} \tag{10}$$

where $\Delta w_p^{(k)}$ is the differential frequency shifts between the receivers of the pth pair during the kth observation period. We assume negligible change in array-source geometry during each observation period; thus the components of $\Delta \mathbf{w}$ are essentially time invariant. A more general approach which takes into account the time variations of the differential Doppler shifts (thus large-scale change in array-source geometry even during a single observation period) is discussed in [1].

The subset $\{\Delta w_p^{(k)}\}_{p=1}^{m-1}$ of differential Doppler shifts is uniquely determined by $\mathrm{s}^{(k)}$, the vector of track parameters which characterizes source trajectory during the kth observation period. Furthermore, the components of $\mathbf{s}^{(k)}$ can be expressed in terms of the components of $\mathbf{s}^{(n)}$, the desired set of track parameters, using the interpolation formulas (8a)-(8b). The various elements of $\Delta\mathbf{w}$ are therefore functionals of $\mathbf{s}^{(n)}$ and one can write formally

$$\Delta\mathbf{w} = G(\mathbf{s}^{(n)}) . \tag{11}$$

Conversley, if a sufficient set of differential Doppler shifts is known, the $\mathbf{s}^{(n)}$ components are uniquely determined by the $\Delta\mathbf{w}$ components. For the moment we only point out that the number of required differential Doppler parameters must be at least equal to the number of unknown track parameters. Increases in the number of $\Delta\mathbf{w}$ components above this minimum improves trajectory estimates. Reduction in the number of $\Delta\mathbf{w}$ components below the minimum leads to a singular problem and hence complete failure of track estimation.

In practice, a measurement of $\Delta\mathbf{w}$ contains a random error due to the additive noise component. A complete analysis of the measurement procedure is given in [2]. Here we assume spatially incoherent and spectrally white noise (over the receiver frequency band).[1] In that case, errors in measurements of the various $\Delta\mathbf{w}$ components are statistically independent, each of which assumes the mean square value.

$$E[(\Delta\hat{w}_p^{(k)} - \Delta w_p^{(k)})^2] = (12/T_k^3)\{[A_p^{(k)^2}/N_p^{(k)}]^{-1} + [A_m^{(k)^2}/N_m^{(k)}]^{-1}\} \tag{12}$$

where $\Delta\hat{w}_p^{(k)}$ is the estimated value of $\Delta w_p^{(k)}$. $E[\,]$ stands for the statistical expectation of the bracketed quantity, T_k denotes the kth observation period. $N_p^{(k)}$ and $A_p^{(k)}$ are, respectively, the noise spectral level and the signal amplitude at the pth receiver during that time period. For sinusoidal signals, the receiver frequency band is determined by the inverse of the observation period. Thus the quantity

$$\mathrm{SNR}_p^{(k)} = A_p^{(k)^2}/N_p^{(k)}\, T_k^{-1} \tag{13}$$

is the signal-to-noise power ratio at the pth receiver during T_k.

The estimation problem can now be stated as follows: Given the vector measurement $\Delta\hat{\mathbf{w}}$, find the minimum mean square estimate $\hat{\mathbf{s}}^{(n)}$ of $\mathbf{s}^{(n)}$. In other words, $\hat{\mathbf{s}}^{(n)}$ is that value of $\mathbf{s}^{(n)}$ which minimizes the quantity

$$[\Delta\hat{\mathbf{w}} - G(\mathbf{s}^{(n)})]^{\mathrm{T}} \operatorname{cov}^{-1}(\Delta\hat{\mathbf{w}})\, [\Delta\hat{\mathbf{w}} - G(\mathbf{s}^{(n)})] \tag{14}$$

where $\operatorname{cov}(\Delta\hat{\mathbf{w}})$ is the measurement error covariance matrix. In light of the previous discussion, it assumes a diagonal form with the diagonal elements given by (12).

To simplify the form of (14), we assume equal signal-to-noise ratio conditions (SNR) for all k and p combinations in (13). For equal measurement periods (T), $\operatorname{cov}(\Delta\hat{\mathbf{w}})$ assumes the simplified form

$$\operatorname{cov}(\Delta\hat{\mathbf{w}}) = (24/T^2\ \mathrm{SNR})\, I . \tag{15}$$

With this convention $\hat{\mathbf{s}}^{(n)}$, the estimate of the projectile position at $t = t_n$, is obtained by minimizing the norm of the vector $[\Delta\hat{\mathbf{w}} - G(\mathbf{s}^{(n)})]$, i.e.,

$$\min_{\mathbf{s}^{(n)}} \{[\Delta\hat{\mathbf{w}} - G(\mathbf{s}^{(n)})]^{\mathrm{T}} [\Delta\hat{\mathbf{w}} - G(\mathbf{s}^{(n)})]\} \rightarrow \hat{\mathbf{s}}^{(n)} . \tag{16}$$

The authors have successfully used the iterative algorithm described in [3], which is available as subroutine ZXSSQ in the International Mathematical and Statistical Library (IMSL).

With regard to the question of the initial guess required for such an iterative algorithm, it should be recalled that the a priori information on the expected trajectory of the projectile is rather accurate. Thus the expected target characterization could serve as the initial guess.

Estimation Performance

In the previous section we have discussed the procedure for estimating the track parameters at t_n by utilizing differential Doppler shifts measured at various periods t_k along the track. This section presents a technique for evaluating the performance of that estimation procedure and demonstrates its optimality.

In order to estimate performance, let us assume that our estimate $\hat{\mathbf{s}}^{(n)}$ is in the neighborhood of the true value $\mathbf{s}^{(n)}$. One can therefore write, to a very good approximation,

$$G(\hat{\mathbf{s}}^{(n)}) = G(\mathbf{s}^{(n)}) + D(\hat{\mathbf{s}}^{(n)} - \mathbf{s}^{(n)}) \tag{17}$$

where D is the matrix of partial derivatives

$$D = [\partial \Delta w_p^{(k)} / \partial s_i^{(n)}] . \tag{18}$$

An explicit description of D is given in the Appendix. Substitution of (17) in (16) results in a linear-weighted least-square problem whose solution is [4].[2]

$$\hat{\mathbf{s}}^{(n)} - \mathbf{s}^{(n)} = (D^{\mathrm{T}}D)^{-1}D^{\mathrm{T}}(\Delta\hat{\mathbf{w}} - \Delta\mathbf{w}) \tag{19}$$

[1] If the receivers are separated by a sufficient distance so that there can be significant differential Doppler shifts, then the assumption of spatially incoherent (statistically independent from receiver to receiver) noise is likely to be very good (an exception would be noise contributed by an interfering point source).

[2] Equation (19) may describe a linear equation for small additive correction in the estimated value. In that case, $\mathbf{s}^{(n)}$ represents a trial solution of the minimization problem of (16), and $\hat{\mathbf{s}}^{(n)}$ represents the iterated value. The foregoing process can be repeated until the correction is negligible.

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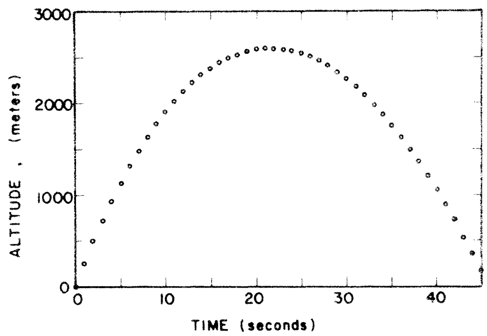

Fig. 2. Source altitude versus time in numerical example.

where $\Delta w = G(s^{(n)})$. Taking the second-order moment on both sides of (19), one obtains

$$\mathrm{cov}(s^{(n)}) = (D^T D)^{-1} D^T \mathrm{cov}(\Delta\hat{w}) D (D^T D)^{-1}$$

$$= (24/T^2 \mathrm{SNR}) (D^T D)^{-1} . \tag{20}$$

In the second version of (20) we have used the simplified form of $\mathrm{cov}(\Delta\hat{w})$ given in (15).

Successive substitutions of (A1) and (A4) in (20) yield a final form of $\mathrm{cov}(\hat{s}^{(n)})$,

$$\mathrm{cov}(\hat{s}^{(n)}) = (24/T^2 \mathrm{SNR}) [\sum_k D^{(k)^T} D^{(k)}]^{-1}$$

$$= (24/T^2 \mathrm{SNR}) \{ \sum_k [P(t_k)\Phi(t_n, t_k)]^T \cdot [P(t_k)\Phi(t_n, t_k)] \}^{-1} \tag{21}$$

where P and Φ are defined throughout (A5)-(A7) and (A9)-(A12), respectively, and k sums over all time periods in which measurements are available.

Discussion

Our estimation procedure utilizes a nonlinear least-square algorithm with simultaneous processing of all the data jointly. Yet a computationally more attractive procedure is to use a recursive approach, i.e., to obtain an initial estimate of $s^{(n)}$ from only a minimal set of differential measurements (= 6) and then to update the estimate whenever additional measurements become available. In this situation, there is no need to store past measurements, and the set simultaneous nonlinear equations required in the estimation process is reduced. If the initial estimate is in the neighborhood of the true value, one can linearize the updating matrix (the so-called extended Kalman filter), as was done in (19), to reduce computational complexity.

Our objective was to minimize the mean-square estimation error. Toward that objective the recursive procedure is likely to be suboptimal. On the other hand, one can show that the error covariance matrix given by (20) or (21) is, in fact, the absolute lower bound given by the inverse of the Fisher information matrix [5]. Thus simultaneous processing of all the received data jointly clearly results in a minimum mean-square error estimate of $s^{(n)}$.

The number of $\Delta\hat{w}$ components required for the estimation procedure must be at least equal to the number of unknown track parameters. Practically it may be difficult to find seven receiver locations within the limited area of the array-yielding frequency measurements which are significantly different from one another. Thus differential Doppler measurements from a single observation period may not be sufficient for the localization procedure. However, one can use the time evolution of differential Doppler measurements observed at one receiver pair to characterize the trajectory (up to an ambiguity due to symmetry). Additional receivers improve localization accuracy but are not indispensable for the localization procedure.

An important practical question is therefore the determination of a set $\Delta\hat{w}$ large enough to obtain near-optimal performance but small enough to keep problems of measurement and computation within reasonable bounds. The following numerical example illustrates the preceding considerations.

Numerical Example

The trajectory chosen for the example is determined by the initial track parameters

$$s(o) = (-4544.6, -1221.6, 0, 62.76, 172.43, 262.0)^T.$$

The first three entries are measured in meters, the last three entries in meters per second. The trajectory is obtained by integration of the equations of motion (with $B = 10^{-4} \mathrm{m}^{-1}$). Using an integration step of $\Delta t = 1$ s yields $z(t)$, as shown in Fig. 2. Since $z = 0$ was crossed during the forty-sixth second of flight, the track parameters were estimated at $t_n = 45$ s. For reference, the actual set is

$$s(45) = (-2558.23, 4235.87, 62.18, 31.34, 86.12, -196.73)^T$$

Three receivers were used, located at (0, 0, 0), $(-b, 0, 0)$, and $(-b, -b, 0)$, where $b = 100$ m.

Fig. 3 shows the normalized rms error in the estimate of the x-coordinate of source position at $t = 45$ s [given by the square root of the (1, 1) element of (21)]. First a minimal set of three measurement periods were used, taken at the forty-fifth, forty-fourth, and forty-third seconds of the flight. Curve a describes the evolution of the estimation error as one adds additional measurements going backward one second at a time. Thus the horizontal coordinate determines the earliest time of the flight in which measurements are available. One can clearly see that the improvement in accuracy with additional measurements (given by the slope) is more significant initially.

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Curve b in Fig. 3 gives the error when measurements are added every two seconds along the flight. The duration of each measurement period could now be twice as long as that used in curve a, while the error difference between the curves is smaller than a factor of $2^{3/2}$. As the error is linearly related to $T^{-3/2}$ (through SNR), the conclusion from the above is that fewer but longer measurement periods improve estimation accuracy. Note, however, that the frequency measurements used in the estimation process contain bias terms resulting from the assumption that there is a negligible change in array-source geometry during each measurement period. The magnitude of this bias increases as T increases.

Both curves c and d are due to a minimal set of three equally spaced measurements. In c the measurement periods are spread at $t = 45, 45 - i$, and $45 - 2i$. In d the three measurements are consecutive, 1 second apart. Both errors are given as a function of the time of the first measurement period. It is interesting to note that the separated periods (curve c) yielded significantly smaller error than the consecutive periods (curve d). The two singular points in curve d indicate that in some parts of the trajectory, the minimal set of consecutive measurements is insufficient.

The numerical example was repeated for twice as long a baseline ($b = 200$). On the average the error, $\sigma(x)$, was reduced by a factor of 2. In order to obtain actual values of the rms error one must scale Fig. 3 by the normalizing constant $\lambda/T(\mathrm{SNR})^{1/2}$. The right vertical axis of Fig. 3 was scaled for SNR = 1000, $\lambda = 0.33$ m, and $T = 1$s. The scaling factor could be rewritten as $c/fT(\mathrm{SNR})^{1/2}$ where c is the velocity of propagation and f is the radiated frequency. It is interesting to note that the most critical feature in this scale factor is fT, the total number of "carrier" cycles in the observation interval.

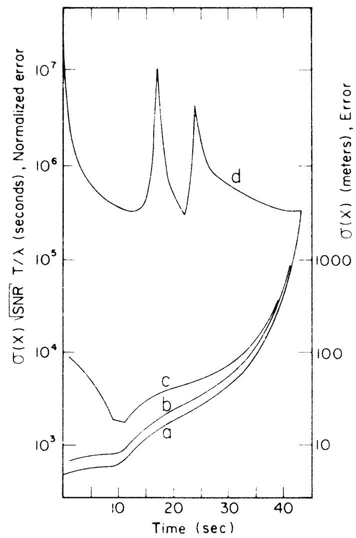

Fig. 3. Rms position error in x direction versus time of first measurement. Measurements taken every (a) 1 s; (b) 2 s; (c) $(45 - t)/2$ s; and (d) 1 s (only 3 measurement periods in (c) and (d).

Bias Error

Our discussion thus far has dealt with the random error in $\hat{\mathbf{s}}^{(n)}$ resulting from the random error in the measurement of $\Delta\mathbf{w}$. Among the sources for bias error are the noninfinitesimal interpolation step, Δt, and the difference between our rather simple ballistic model and the true motion of the projectile. The first source of bias error can be considerably reduced by replacing the linear interpolation formulas given in set (8) with higher order formulas

$$\mathbf{x}_{k-1} = \mathbf{x}_k - \Delta t\, \dot{\mathbf{x}}_k + (1/2)(\Delta t)^2\, \ddot{\mathbf{x}}_k \tag{22A}$$

$$\dot{\mathbf{x}}_{k-1} = \dot{\mathbf{x}}_k - \Delta t\, \ddot{\mathbf{x}}_k + (1/2)(\Delta t)^2\, [(\partial \ddot{\mathbf{x}}_k/\partial \mathbf{x}_k)\, \dot{\mathbf{x}}_k + (\partial \ddot{\mathbf{x}}_k/\partial \dot{\mathbf{x}}_k)\, \ddot{\mathbf{x}}_k\,] \tag{22B}$$

where the two partial derivative matrices are as given in (A11) and (A12).

Without in-depth discussion of more detailed ballistic models, our discussion of the second source of bias error has to remain qualitative. Intuitively, it seems that the further along the trajectory the measurement is from the point of interest (e.g., impact), the larger is its contribution to bias error. Thus, while Fig. 3 (curve a) indicates that adding earlier measurements always reduces the random error, bias error considerations are likely to show that beyond a certain point, adding earlier measurements will increase the bias error, and maybe the overall error. An indication to that effect was obtained from the numerical example above, when the ballistic constant used in the estimation process was 25 percent higher than the one used in generating the trajectory and the measurement vector $\Delta\hat{\mathbf{w}}$. In that case, the minimum bias error at impact was achieved when the measurements started at $t = 35$ s. Adding earlier measurements increased the bias error.

Such bias error considerations indicate that the a priori information on the trajectory, derived from initial position and muzzle velocity, could not add significantly to the estimation procedure, except for providing the first guess.

Summary and Conclusions

Passive tracking, which is widely used in sonar, is adapted here to an electromagnetic application of a projectile carrying a CW source. The availability of the source and the fairly well-known model of the projectile motion are combined to provide a feasible tracking technique. Where applicable, this technique can replace costly active radar tracking.

A least-square estimation procedure was suggested and outlined in detail. Furthermore, we have given a performance analysis for evaluating the lower bound on the random error in the estimation due to the additive

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noise component. The performance analysis was applied to a numerical example in order to show the relative estimation error for various combinations of measuring periods along the track. Bias error was briefly discussed, and together with implementation techniques, are the topics for ongoing work.

Appendix

Partial Derivative Matrix D

The matrix D can be decoupled as follows:

$$D = (D^{(1)}, D^{(2)}, D^{(3)}, \ldots)^{\mathrm{T}} . \tag{A1}$$

The $(m-1) \times 6$ submatrix $D^{(k)}$ is associated with the kth measurement period. Its elements are given by

$$D^{(k)}_{pi} = \partial \Delta w^{(k)}_p / \partial s^{(n)}_i \tag{A2}$$

where $s^{(n)}_i$ is the ith component of $\mathbf{s}^{(n)}$. Making use of the chain rule

$$\partial \Delta w^{(k)}_p / \partial s^{(n)}_i = \sum_{\ell=1}^{6} \{[\partial \Delta w^{(k)}_p / \partial s^{(k)}_\ell] [\partial s^{(k)}_\ell / \partial s^{(n)}_i]\} \tag{A3}$$

$D^{(k)}$ can be further decoupled into the form

$$D^{(k)} = P(t_k) \Phi(t_n, t_k) \tag{A4}$$

where $P(t_k)$ is the matrix whose elements are

$$P(t_k)_{pi} = \partial \Delta w^{(k)}_p / \partial s^{(k)}_i = (2\pi/\lambda_k)(\partial \dot{r}^{(k)}_m / \partial s^{(k)}_i - \partial \dot{r}^{(k)}_p / \partial s^{(k)}_i) . \tag{A5}$$

In the second version of (A5) we used (2). λ_k is the radiated wavelength during the kth observation period. The derivatives called for in (A5) are obtained from (4)

$$\partial \dot{r}^{(k)}_p / \partial x_k = \{[(y_k - g_p)^2 + (z_k - h_p)^2]\, \dot{x}_k - (x_k - f_p)[(y_k - g_p)\, \dot{y}_k + (z_k - h_p)\, \dot{z}_k]\} / [(x_k - f_p)^2 + (y_k - g_p)^2 + (z_k - h_p)^2]^{3/2} \tag{A6}$$

$$\partial \dot{r}^{(k)}_p / \partial \dot{x}_k = (x_k - f_p) / [(x_k - f_p)^2 + (y_k - g_p)^2 + (z_k - h_p)^2]^{1/2} . \tag{A7}$$

In complete symmetry one can write down immediately the derivatives with respect to y, $\dot{y}$, z, and $\dot{z}$.

The second matrix term $\Phi(t_n, t_k)$ appearing in (A4) characterizes the transition from t_k to t_n. Its (i, j) element is given by

$$\Phi(t_n, t_k)_{ij} = \partial s^{(k)}_i / \partial s^{(n)}_j . \tag{A8}$$

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Using the chain rule successively one immediately obtains

$$\Phi(t_n, t_k) = \prod_{i=1}^{n-k} \Phi(t_k + i\Delta t, t_k + (i-1)\Delta t) \tag{A9}$$

where $t_n = t_k + (n-k)\Delta t$. Differentiation of the interpolation formulas (8) results in the basic element of the matrix multiplication

$$\Phi(t_k, t_k - \Delta t) = \left| \begin{array}{c|c} I & -\Delta t\, I \\ \hline -\Delta t(\partial \ddot{\mathbf{x}}_k / \partial \mathbf{x}_k) & I - \Delta t(\partial \ddot{\mathbf{x}}_k / \partial \dot{\mathbf{x}}_k) \end{array} \right| \tag{A10}$$

where

$$\partial \ddot{\mathbf{x}}_k / \partial \mathbf{x}_k = L B e^{-L z_k} |\dot{\mathbf{x}}_k| \left| \mathbf{0} \,\vdots\, \mathbf{0} \,\vdots\, \dot{\mathbf{x}}_k \right| \tag{A11}$$

and

$$\partial \ddot{\mathbf{x}}_k / \partial \dot{\mathbf{x}}_k = -B e^{-L z_k} (|\dot{\mathbf{x}}_k| \, I + \dot{\mathbf{x}}_k \dot{\mathbf{x}}_k^{\mathrm{T}} / |\dot{\mathbf{x}}_k|) \tag{A12}$$

where $\mathbf{0}$ is the (3×1) zero vector. $|\dot{\mathbf{x}}_k|$ is the norm (magnitude) of the velocity vector $\dot{\mathbf{x}}_k$ given by

$$|\dot{\mathbf{x}}_k| = (\dot{x}_k^2 + \dot{y}_k^2 + \dot{z}_k^2)^{1/2} . \tag{A13}$$

EHUD WEINSTEIN
Woods Hole Oceanographic Inst.
Woods Hole, MA 02543

NADAV LEVANON
Dept. of Electronic Systems
Tel-Aviv University
Ramat-Aviv, Tel-Aviv, 69978, Israel

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