DCT. In terms of its application, using CMT as the base, both transfrom and hybrid coding [12] of broadcast quality TV signal for digital transmission at reduced bit rates is being investigated.

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# Analytic Inversion of Fisher's Information Matrix for Delay, Delay Rate, and Higher Derivatives

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*Abstract*—**Fisher's information matrix for delay (which corresponds to range), Doppler (range rate), and higher derivatives was recently presented by Schultheiss and Weinstein [1], [2]. Its inverse, which provides a lower bound on the error covariance matrix, was obtained numerically by them. An analytic inversion is presented here. It applies to the case of one reference signal (single-frequency) and one received signal, delayed and contaminated by white noise. The analytic inversion provides an insight on the increases in the variances of delay and Doppler estimations in the presence of higher derivatives. Asymptotically, the delay variance increases linearly with the highest derivative order, and the Doppler variance increases cubically.**

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## I. INTRODUCTION

Schultheiss and Weinstein [1], [2] have obtained Fisher's information matrix for the unbiased estimation of delay and its derivatives, of which the first one corresponds to the Doppler angular frequency. They chose a polynomial representation of the delay

$$D(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_q t^q + \cdots + a_p t^p \tag{1}$$

in which the coefficients are related to the delay derivatives by

$$\frac{d^q}{dt^q}\left[D(t)\right]_{t=0} = q!\, a_q. \tag{2}$$

In limiting the polynomial order to $p$, one assumes that the highest existing derivative is the $p$th derivative.

For the simple case when the delay[1] is measured using a sinusoidal signal, a single receiver, and a noise-free reference signal, Fisher's matrix for the rearranged coefficients vector

$$\boldsymbol{a} = (a_0, a_2, a_4, \cdots, a_1, a_3, a_5, \cdots)^T \tag{3}$$

is given by [1]

$$J(\boldsymbol{a}) = \frac{A^2 T}{N_0}\left[\begin{array}{c|c} J^e & 0 \\ \hline 0 & J^o \end{array}\right] \tag{4}$$

where the even coefficients matrix is

$$J^e = \begin{bmatrix} 1 & \frac{(T/2)^2}{3} & \frac{(T/2)^4}{5} & \cdots \\ \frac{(T/2)^2}{3} & \frac{(T/2)^4}{5} & \frac{(T/2)^6}{7} & \cdots \\ \frac{(T/2)^4}{5} & \frac{(T/2)^6}{7} & \frac{(T/2)^8}{9} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = [J^e]_{i,j=1\cdots r} \tag{5}$$

$$r = \begin{cases} \dfrac{p}{2} + 1, & p \text{ even} \\[2ex] \dfrac{p}{2} + \dfrac{1}{2}, & p \text{ odd} \end{cases} \tag{6}$$

and the odd coefficients matrix is

$$J^o = \begin{bmatrix} \frac{(T/2)^2}{3} & \frac{(T/2)^4}{5} & \frac{(T/2)^6}{7} & \cdots \\ \frac{(T/2)^4}{5} & \frac{(T/2)^6}{7} & \frac{(T/2)^8}{9} & \cdots \\ \frac{(T/2)^6}{7} & \frac{(T/2)^8}{9} & \frac{(T/2)^{10}}{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = [J^o]_{m,n=1\cdots s} \tag{7}$$

[1] $D$ here is phase delay in radians. To convert to time delay, divide $D$ by $2\pi f$.

$$s = \begin{cases} \dfrac{p}{2}, & p \text{ even} \\ \dfrac{p}{2} + \dfrac{1}{2}, & p \text{ odd.} \end{cases} \tag{8}$$

$N_0$ is the noise spectral energy, $A$ is the signal amplitude, and $T$ is the total observation time. Note that the expression preceding the matrix in (4) is the ratio of twice the signal energy during the observation time to the noise energy. It is this expression that changes with different signals or receiving systems.

The inverse of (4) provides a lower bound on the error covariance matrix of $\boldsymbol{a}$. Numerical inversions for several polynomial orders were given in [1] and [2]. In the following section, we will outline the computation of an analytic inversion of (4), following the approach of Proschan [3], and present the final result.

## II. The Inverse Matrix

We first note that $J^e$ can be factored as

$$J^e = B^e K^e B^e \tag{9}$$

where

$$B^e_{i,j} = \begin{cases} \left(\dfrac{T}{2}\right)^{2(i-1)}, & i = j \\ 0, & i \neq j \end{cases} \tag{10}$$

and

$$K^e_{i,j} = \frac{1}{2(i+j) - 3}. \tag{11}$$

Hence, the inverse of $J^e$ is given by

$$(J^e)^{-1} = (B^e)^{-1} (K^e)^{-1} (B^e)^{-1}. \tag{12}$$

The inverse of the diagonal matrix $B^e$ is the inverse of each of its elements, so we are left only with the task of inverting $K^e$.

Similarly, we note that

$$J^o = B^o K^o B^o \tag{13}$$

where

$$B^o_{m,n} = \begin{cases} \left(\dfrac{T}{2}\right)^{2m-1}, & m = n \\ 0, & m \neq n \end{cases} \tag{14}$$

and

$$K^o_{m,n} = \frac{1}{2(m+n) - 1}. \tag{15}$$

The inversions of $K^e$ and $K^o$ are performed using the Cauchy identities for determinants of "alternating functions" [4]

$$\left|\frac{1}{x_i - y_j}\right|_{i,j=1,2,\cdots r} = (-1)^{(1/2)r(r-1)} \cdot \frac{\prod_{j>i} (x_j - x_i) \prod_{j>i} (y_j - y_i)}{\prod_{i=1}^{r} \prod_{j=1}^{r} (x_i - y_j)} \tag{16}$$

$$\left|\frac{1}{x_i - y_j}\right|_{i \neq p, j \neq q} = (-1)^{(1/2)(r-1)(r-2)} \cdot \frac{\prod_{\substack{j>i \\ i \neq p, j \neq p}} (x_j - x_i) \prod_{\substack{j>i \\ i \neq q, j \neq q}} (y_j - y_i)}{\prod_{i \neq p} \prod_{j \neq q} (x_i - y_j)}. \tag{17}$$

Note from (11) that we can write $K^e$ in the form of an alternating function

$$K^e_{i,j} = \frac{1}{x_i - y_j} \tag{18}$$

if we choose

$$x_i = 2i \tag{19}$$

and

$$y_j = 3 - 2j. \tag{20}$$

Similarly, we can write

$$K^o_{m,n} = \frac{1}{x_m - y_n} \tag{21}$$

if we choose

$$x_m = 2m \tag{22}$$

and

$$y_n = 1 - 2n. \tag{23}$$

Note that (17) is the $p, q$ cofactor of the matrix whose determinant appears in (16). Hence, the $p, q$ element of the inverse matrix is obtained by dividing (17) by (16). Repeating this for all the elements yields the inverse of (4)

$$[J(\boldsymbol{a})]^{-1} = \frac{N_0}{A^2 T} \left[\begin{array}{c|c} (J^e)^{-1} & 0 \\ \hline 0 & (J^o)^{-1} \end{array}\right] \tag{24}$$

where

$$(J^e)^{-1}{}_{i,j=1\cdots r} = \frac{(-1)^{i+j}}{2^{4r-2}(2i+2j-3)} \left(\frac{2}{T}\right)^{2(i+j-2)} \alpha_i \alpha_j \tag{25}$$

with

$$\alpha_i = \frac{[2(r+i-1)]!}{(r+i-1)!\,(r-i)!\,(2i-2)!} \tag{26}$$

and where

$$(J^o)^{-1}{}_{m,n=1\cdots s} = \frac{(-1)^{m+n}}{2^{4s}(2m+2n-1)} \left(\frac{2}{T}\right)^{2(m+n-1)} \beta_m \beta_n \tag{27}$$

with

$$\beta_m = \frac{[2(s+m)]!}{(s+m)!\,(s-m)!\,(2m-1)!}. \tag{28}$$

## III. Main Diagonal Terms

The terms of the main diagonal are of particular interest since they provide a lower bound on the variance of each polynomial coefficient. As was shown in [5], a unified expression for the $\hat{a}_q$ coefficient is given by

$$\operatorname{Var} \hat{a}_q \geqslant \frac{N_0}{A^2 T}\left(\frac{2}{T}\right)^{2q}\left[\frac{(s+q+1)!}{2^s\left(\dfrac{s+q+1}{2}\right)!\left(\dfrac{s-q-1}{2}\right)!\,q!}\right]^2 \cdot \frac{1}{2q+1} \tag{29}$$

where

$$s=\begin{cases} p, & p+q=\text{odd} \\ p+1, & p+q=\text{even}. \end{cases} \tag{30}$$

Of special practical interest are $\operatorname{Var} \hat{a}_0$ and $\operatorname{Var} \hat{a}_1$ which correspond to delay and Doppler, respectively. Setting $q = 0$ and $q = 1$ in (29), we obtain

$$\operatorname{Var} \hat{a}_0 \geqslant \frac{N_0}{A^2 T}\left\{\frac{s!}{2^{s-1}\left[\left(\dfrac{s-1}{2}\right)!\right]^2}\right\}^2 \tag{31}$$

where

$$s=\begin{cases} p, & p \text{ odd} \\ p+1, & p \text{ even} \end{cases} \tag{32}$$

and

$$\operatorname{Var} \hat{a}_1 \geqslant \frac{N_0}{A^2 T}\left(\frac{2}{T}\right)^2\left\{\frac{(s+1)!\,s}{2^s\left[\left(\dfrac{s}{2}\right)!\right]^2}\right\}^2 \frac{1}{3} \tag{33}$$

where

$$s=\begin{cases} p+1, & p \text{ odd} \\ p, & p \text{ even}. \end{cases} \tag{34}$$

When the motion can be represented by a fixed range, the smallest polynomial for an unbiased estimate of the delay is of the zero order. Thus, from (31) and (32) we obtain

$$\operatorname{Var} \hat{a}_0 \big|_{p=0} \geqslant \frac{N_0}{A^2 T}. \tag{35}$$

Note that increasing the polynomial order from zero to one does not effect the variance $\hat{a}_0$.

When the motion results in range rate (Doppler), the smallest polynomial allowing its estimate is of the first order. Thus, from (33) and (34) we obtain

$$\operatorname{Var} \hat{a}_1 \big|_{p=1} \geqslant \frac{N_0}{A^2 T} \frac{12}{T^2}. \tag{36}$$

Both (35) and (36) are well-known results [6].

## IV. Asymptotic Expressions

When unbiased estimates of delay and Doppler are calculated simultaneously with many higher derivatives, their variances [(31) and (33)] approach the asymptotic expressions

$$\operatorname{Var} \hat{a}_0 \big|_{s \gg 1} \gtrsim \frac{N_0}{A^2 T} \frac{2s}{\pi} \tag{37}$$

where $s$ is related to $p$ as in (32), and

$$\operatorname{Var} \hat{a}_1 \big|_{s \gg 1} \gtrsim \frac{N_0}{A^2 T}\left(\frac{2}{T}\right)^2 \frac{2s^2(s+1)}{3\pi} \tag{38}$$

where $s$ is related to $p$ as in (34).

Equations (37) and (38) provide an answer to the question left open in [1]: How do the variances of delay and Doppler increase with the addition of arbitrary large polynomial order? Equations (37) and (38) show (asymptotically) that the delay variance increases linearily with the polynomial order, and the Doppler variance increases cubically.

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# A Two-Parameter Class of Bessel Weightings for Spectral Analysis or Array Processing—The Ideal Weighting-Window Pairs

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***Abstract*—A unified theory for array processing in 1, 2, or 3 dimensions is pointed out and illustrated by a two-parameter class of Bessel weightings. This class subsumes the $I_o$-sinh weighting-window pair as well as the ideal space factor of van der Maas in one dimension. The weightings that realize the ideal space factor in 2 and 3 dimensions are generalized functions more singular than the delta function required in 1 dimension.**

## I. Introduction

A wide variety of time-domain weightings for spectral analysis, whose frequency-domain windows have very good sidelobe behavior, are available in [1], [2]. However, most of the weighting-window pairs in [1], [2] have no parameters in their design equations; that is, the windows are fixed and cannot be altered, as for example, in the Hanning and Hamming windows. A few windows, such as the Dolph–Chebyshev and $I_o$-sinh [3], [4], do have a single parameter in their design equations that allows for a tradeoff between the mainlobe width and the ratio of mainlobe-to-peak-sidelobe. However, the latter have no control over the rate of decay of the sidelobes, the Dolph–Chebyshev case having no decay, and the Kaiser–Bessel case a 6 dB/octave decay. It is obvious that in order to control *both*

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