# Order statistics CFAR for Weibull background

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*Indexing terms: Radar, Algorithms, Errors, Clutter*

**Abstract:** The paper shows how to adopt Rohling's order statistics CFAR algorithm, developed for a Rayleigh background, to the case of a Weibull background with a known shape parameter. When the shape parameter is unknown, Weber and Haykin have proposed a two-parameter algorithm in which the threshold is obtained from two ranked background samples. In the paper the CFAR loss of the Weber–Haykin CFAR algorithm is analysed and compared with another CFAR processor for Weibull clutter suggested by Hansen. It is found that, with the optimal choice of representative ranks, the Weber–Haykin CFAR processor yields a smaller, but still very excessive, detection loss. Finally it is shown that CFAR is maintained when more than two ranked samples are used in setting the threshold, but without improvement in performance.

## 1 Introduction

Rohling [1] has proposed a CFAR algorithm based on order statistics (OS) that exhibits reduced sensitivity to spurious targets. In Rohling's original OS-CFAR algorithm, $M$ background (amplitude) samples are ranked in an increasing order

$$X_1 \leqslant X_2 \leqslant \cdots \leqslant X_i \leqslant \cdots \leqslant X_j \leqslant \cdots \leqslant X_M \quad (1)$$

and squared in a square-law detector yielding

$$z_i = X_i^2 \quad (2)$$

The threshold $T_z$ is obtained by multiplying the $k$th ranked squared-sample (which we term the representative rank) by a scale factor $\alpha_k$:

$$T_z = \alpha_k z_k \quad (3)$$

When the background amplitude samples $\{X_i\}$ are independent, identically distributed (IID) random variables with a Rayleigh probability density function (PDF), Rohling has shown that the relation between the false alarm probability $P_{FA}$ and the scale factor $\alpha_k$ is given by

$$P_{FA} = \frac{M!(\alpha_k + M - k)!}{(M-k)!(\alpha_k + M)!} \quad (4)$$

where $k$ is the representative rank and $M$ is the total number of background samples. Since eqn. 4 is not a function of the scale parameter of the Rayleigh PDF, the algorithm is CFAR.

Paper 7220F (E15), first received 9th June and in revised form 31st October 1989

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We shall show that if the amplitude samples $\{X_i\}$ are Weibull distributed with a *known* shape parameter $C$, then Rohling's result, repeated in eqn. 4, still applies, and CFAR is maintained by his original algorithm if the threshold is set at

$$T_z = \alpha_k^{2/C} z_k \quad (5)$$

Weber and Haykin [2] showed that when the shape parameter is *unknown*, a CFAR algorithm can be obtained if two representative ranked samples, $z_i$ and $z_j$, are used, and the threshold is adaptively set at

$$T_z = z_i^{1-\beta} z_j^{\beta} \quad (6)$$

The coefficient $\beta$ is a function of $M$, $i$, $j$ and the $P_{FA}$. For Weibull distributed samples, Weber and Haykin provided an integral relationship between $P_{FA}$ and $\beta$ that requires a numerical solution. Since a Weibull distributed $X$ yields a Weibull distributed $z$ with different parameters, and since the CFAR algorithm is insensitive to changes in the parameters, the same $\beta$ applies to both linear and square-law detectors. Hence, a threshold which follows a linear detector should be set at

$$T_X = X_i^{1-\beta} X_j^{\beta} \quad (7)$$

The algorithm suggested by Weber and Haykin effectively estimates the shape parameter $C$ from the ratio $z_j/z_i$ and utilises it in eqn. 5 choosing $k = i$. The CFAR loss increases with the variance of the estimated $C$. We quote an analytic expression for the variance of $C$ as a function of the choice of the two ranks, and present analytic results for the detection loss in the special case when $C = 2$. We found that the optimal choice of ranks yields (for $M = 16$ and $P_{FA} = 10^{-5}$) a CFAR loss of 9.4 dB compared to a 2.3 dB loss of a single parameter OS-CFAR. When $M$ is increased to 32 cells the two-parameter CFAR loss drops to 4.2 dB.

We show that processors based on more than two ranks are also CFAR, but have no advantage over the two-ranks processor. We also present a simple approximate expression for $\beta$ that can replace the cumbersome integral equation when $P_{FA} > 10^{-3}$. Finally, for a case of limited uncertainty in $C$, we compare the loss of the Weber–Haykin algorithm to that of the OS-CFAR with a threshold set to yield the nominal $P_{FA}$ at the lowest $C$.

## 2 OS-CFAR for Weibull background with known shape parameter

Let the background amplitude samples be described by an IID random variable $X$ with a Weibull PDF:

$$p_1(X) = \frac{C}{B}\left(\frac{X}{B}\right)^{C-1} \exp\left[-\left(\frac{X}{B}\right)^C\right] \quad 0 \leqslant X \quad (8)$$

Note that by choosing the shape parameter $C = 2$, the Weibull PDF reduces to the Rayleigh PDF (amplitude).

It is well known that various clutter scenes fit a wider range of the shape parameter. A smaller shape parameter implies a longer tailed PDF. When the OS processor

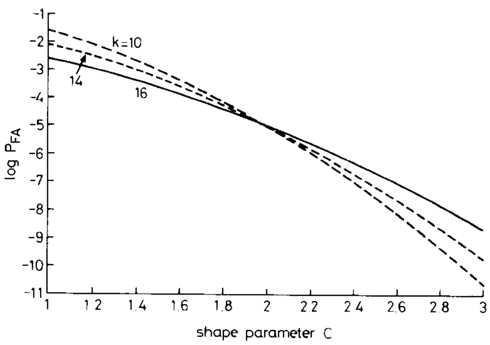


**Fig. 1** *False alarm probability of OS-CFAR in Weibull background as a function of shape parameter*
Nominal design: $P_{FA} = 10^{-5}$ when $C = 2$; $M = 16$

follows a square-law detector, the inputs to the processor depend on $X^2$. Without loss of generality we can normalise with respect to $B$ and obtain the normalised input random variable,

$$z = \left(\frac{X}{B}\right)^2 \tag{9}$$

whose PDF is found from eqn. 8 to be

$$p_2(z) = \frac{C}{2} z^{(C/2)-1} \exp(-z^{C/2}) \tag{10}$$

To be able to use Rohling's analysis we will make an additional substitution and define the random variable $y$,

$$y = z^{C/2} \tag{11}$$

whose PDF is the exponential PDF for which Rohling performed his analysis:

$$p_3(y) = \exp(-y) \tag{12}$$

The threshold for $y$, $T_y$ that would have yielded a given $P_{FA}$ would have been

$$T_y = \alpha_k y_k \tag{13}$$

where $\alpha_k$ is found from eqn. 4. The threshold for $z$, which is the actual threshold to be used, is obtained from $T_y$ by using eqn. 11

$$T_z = T_y^{2/C} \tag{14}$$

which yields

$$T_z = \alpha_k^{2/C} z_k = \gamma_k z_k \tag{15}$$

where

$$\gamma_k = \alpha_k^{2/C} \tag{16}$$

is the actual scale factor that has to be used with a square-law detector. If a linear detector is used, the scale factor should be the square root of the one found above, i.e. the threshold for the original (and not squared) samples $\{X_i\}$ should be

$$T_X = \alpha_k^{1/C} X_k \tag{17}$$

From eqns. 16 and 4 we can obtain a general expression for the false alarm probability, as a function of the actually used scale factor $\gamma_k$, for a square-law OS-CFAR in a Weibull background with any shape parameter $C$:

$$P_{FA} = \frac{M!(\gamma_k^{C/2} + M - k)!}{(M-k)!(\gamma_k^{C/2} + M)!} \tag{18}$$

Eqn. 18 allows us to study the sensitivity of the original OS-CFAR algorithm to changes in the shape parameter. Fig. 1 shows the expected $P_{FA}$ as a function of $C$ when the scale factor was set to yield $P_{FA} = 10^{-5}$ when $C = 2$.

## 3 Two-parameter OS-CFAR using two ordered samples

The above results for the single parameter OS-CFAR, lead us to the format of a two-parameter OS-CFAR. We can still use eqn. 15 or 17 but since there is no *a priori* knowledge of $C$ it must be estimated. Dubey [4] has analysed percentile estimators for Weibull parameters. For the case in which the two parameters must be estimated simultaneously (which is our case) he suggested an estimator for the shape parameter based on two ordered samples $X_i$ and $X_j$. Dubey's estimator for $C$ is given by

$$\hat{C} = \frac{\ln[-\ln(1-h_j)] - \ln[-\ln(1-h_i)]}{\ln X_j - \ln X_i} \tag{19}$$

where

$$h_j = \frac{j}{M+1} \tag{20}$$

Using $\hat{C}$ in place of $C$ in eqn. 17, and selecting $k = i$ will yield the Weber–Haykin threshold

$$T_X = X_i\left(\frac{X_j}{X_i}\right)^\beta = X_i^{1-\beta} X_j^\beta \tag{21}$$

where

$$\beta = \frac{\ln \alpha_i}{\ln[-\ln(1-h_j)] - \ln[-\ln(1-h_i)]} \tag{22}$$

and where $\alpha_i$ is obtained from eqn. 4 using the desired $P_{FA}$.

Exactly the same result would have been obtained for $T_z$ had we started our analysis from eqn. 14 rather than eqn. 17, noticing that the shape parameter of the Weibull PDF in eqn. 10 is $C/2$. In other words, for $T_z$ we will get

$$T_z = z_i\left(\frac{z_j}{z_i}\right)^\beta = z_i^{1-\beta} z_j^\beta \tag{23}$$

where $\beta$ is the same as in eqn. 22. This result should be expected since the PDFs of $X$ and $z$ are both Weibull with different shape parameters, and the CFAR algorithm as expressed in eqn. 21 is independent of the shape parameter.

If a perfect estimate of $C$ could have been obtained (no bias and zero variance), then the $P_{FA}$ reached using eqns. 23 and 22 would have been equal to the nominal $P_{FA}$ from which $\alpha_i$ was calculated. However, as Dubey showed, the estimate is asymptotically normal with the correct mean $C$ and variance

$$\operatorname{var} \hat{C} = \frac{C^2}{Mt^2}\left(\frac{q_i}{t_i^2} + \frac{q_j}{t_j^2} - 2\frac{q_i}{t_i t_j}\right) \tag{24}$$

where

$$q_i = \frac{h_i}{1-h_i}; \quad t_i = -\ln(1-h_i); \quad t = \ln t_i - \ln t_j \tag{25}$$

Dubey [4] found that the smallest variance is obtained when

$$h_i = 0.1673 \quad \text{and} \quad h_j = 0.9737$$

and that the variance will be equal to $0.9163C^2/M$. For $C = 2$ and $M = 16$ we find that the smallest standard deviation of $\hat{C}$ will be 0.49, reached when $i = 3$ and $j = 16$. The standard deviation of $\hat{C}$ as a function of $i$, with $j$ as a parameter, appears in Fig. 2.

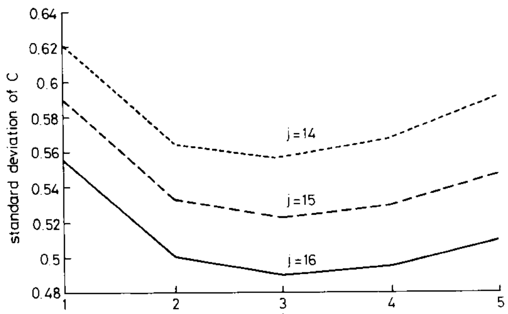


**Fig. 2** *Standard deviation of estimated shape parameter as a function of ranks of two representative samples*
$C = 2$; $M = 16$

We should point out that Dubey also calculated the variance of $\hat{C}$ obtained with a maximum likelihood estimator. For $C = 2$ and $M = 16$ the standard deviation would have been 0.39. This indicates that a better estimator could be found.

A large variance of $\hat{C}$ will result in a different $P_{FA}$. In that case it is necessary to resort to the exact calculation of the $P_{FA}$ as developed in Reference 2. We shall briefly repeat that calculation.

We first note from integrating eqn. 8 that the distribution function of the random variable $X$ (assuming $B = 1$) is given by

$$P_1(X) = 1 - \exp(-X^C) \tag{26}$$

We will designate the value in the $i$th ranked cell as the random variable $x$, and the value in the $j$th ranked cell as the random variable $y$, and maintain that $i < j$. The joint PDF of $X_i$ and $X_j$ will be denoted by $p_{ij}(x, y)$. From Reference 5 we have

$$p_{ij}(x, y) = \begin{cases} Q_2[P(x)]^{i-1}p(x)[P(y) - P(x)]^{j-i-1} & \\ \times\, p(y)[1 - P(y)]^{M-j} & x \leqslant y \\ 0 & \text{elsewhere} \end{cases} \tag{27}$$

where

$$Q_2 = \frac{M!}{(i-1)!(j-i-1)!(M-j)!} \tag{28}$$

If the threshold $T_x$ were to be a predetermined constant, then the false alarm probability would be obtained from eqn. 26 as

$$P_{FA} = 1 - P_1(T_x) = \exp(-T_X^C) \tag{29}$$

However, the threshold is set adaptively according to eqn. 21 and becomes a random variable. Therefore, $P_{FA}$ is obtained by averaging as

$$P_{FA} = \int_{y=0}^{\infty} \int_{x=0}^{y} \exp[-(x^{1-\beta}y^{\beta})^C] p_{ij}(x, y)\, dx\, dy \tag{30}$$

Using eqn. 8 (with $B = 1$) and eqn. 26 in eqn. 27 and eqn. 27 in eqn. 30, and substituting

$$u = x^C \quad v = y^C \tag{31}$$

we get the integral relation between $\beta$ and $P_{FA}$:

$$\begin{aligned} P_{FA} = Q_2 \int_{v=0}^{\infty} \int_{u=0}^{v} & \exp(-u^{1-\beta}v^{\beta}) \\ & \times [1 - \exp(-u)]^{i-1} \exp(-u) \\ & \times [\exp(-v)]^{M-j+1} \\ & \times [\exp(-u) - \exp(-v)]^{j-i-1}\, du\, dv \end{aligned} \tag{32}$$

The small differences between our equation and the one in Reference 2 stem from the fact that our samples are ranked upward ($X_i \leq X_j$) whereas in Reference 2 the ranking is downward.

The disagreement of the two relations between $\beta$ and $P_{FA}$ (the true relation expressed in eqn. 32 and the approximate relation embedded in eqns. 22 and 4) will be demonstrated for the case $M = 16$, $i = 3$ and $j = 16$. For $P_{FA} = 10^{-5}$ eqn. 32 yields $\beta = 2.1337$. Using this value of $\beta$ in eqn. 22 yields $\alpha_3 = 304.72$. Using $\alpha_3$ in eqn. 4 yields $P_{FA} = 1.028 \times 10^{-4}$, which disagrees with the true $P_{FA}$ by a factor of about 10. We found that the disagreement decreases as the $P_{FA}$ is increased.

## 4 CFAR loss of the Weber–Haykin algorithm when $C = 2$

The CFAR loss of the threshold defined in eqn. 6 can be calculated for the special case of a Rayleigh fluctuating target in a Rayleigh background, and when the CFAR processor is preceded by a square-law detector. The Rayleigh background has an amplitude PDF as given in eqn. 8, with shape parameter $C = 2$. The loss calculated for one shape parameter should be a guide to the loss to be expected with other shape parameters. We emphasise that the shape parameter is unknown to the processor, which is why the threshold of eqn. 6 is used to begin with.

For a Rayleigh fluctuating target in a Rayleigh background, the output of a square-law detector $z$ has an exponential PDF:

$$p_4(z) = D \exp(-Dz) \tag{33}$$

where

$$D = \frac{1}{1 + \overline{SNR}} \tag{34}$$

$\overline{SNR}$ being the ratio between the target average power and the background average power. The corresponding distribution function is

$$P_4(z) = 1 - \exp(-Dz) \tag{35}$$

hence the probability of the random variable $z$ exceeding a fixed threshold $T$ is given by

$$P(z \geqslant T) = 1 - P_4(T) = \exp(-DT) \tag{36}$$

In the Weber–Haykin CFAR, $T$ is a random variable defined in eqn. 6. If we denote the value of $z_i$ as $u$ and the value of $z_j$ as $v$, and their joint PDF as $p_{ij}(u, v)$, then the probability of detection will be

$$P_D = \int_{v=0}^{\infty} \int_{u=0}^{v} \exp(-Du^{1-\beta}v^{\beta}) p_{ij}(u, v)\, du\, dv \tag{37}$$

The joint PDF of $u$ and $v$ is obtained by using $u$ and $v$ instead of $x$ and $y$ in eqn. 27 with the parent functions

$p_4(\cdot)$ and $P_4(\cdot)$ as found from eqns. 33 and 35 by setting $D = 1$. This leads to an integral relation between $\beta$, $D$ and $P_D$:

$$P_D = Q_2 \int_{v=0}^{\infty} \int_{u=0}^{v} \exp(-Du^{1-\beta}v^{\beta}) \times [1 - \exp(-u)]^{i-1} \exp(-u) \times [\exp(-v)]^{M-j+1} \times [\exp(-u) - \exp(-v)]^{j-i-1} \, du \, dv \quad (38)$$

Eqn. 38 differs from eqn. 32 only by the term $D$ in the first exponent. Hence, only a minor modification is required in the program which solved eqn. 32 numerically. To remove any doubts, we reiterate that eqn. 32 gives the $P_{FA}$ of the Weber–Haykin CFAR in the case of a Weibull background with any shape parameter $C$. While eqn. 38 gives $P_D$ only when the background shape parameter is $C = 2$, the target is a Rayleigh fluctuating target and the processor follows a square-law detector.

For the situation covered by eqn. 38 we demonstrate the CFAR loss of the Weber–Haykin algorithm for the case of $M = 16$ and $P_{FA} = 10^{-5}$. The two representative ranks are $i = 3$ and $j = 16$ which yield the smallest loss. Solving eqn. 32 with these four parameters yields $\beta = 2.133$. With this value of $\beta$, $P_D$ as a function of the average $SNR$ was calculated using eqn. 38 and the result is represented by the solid line in Fig. 3. For comparison the figure also includes the performances of a fixed threshold detector (dotted line) and of Rohling's original OS-CFAR (dashed line) whose threshold is set according to eqn. 3, with $k = 12$, which yields the smallest loss.

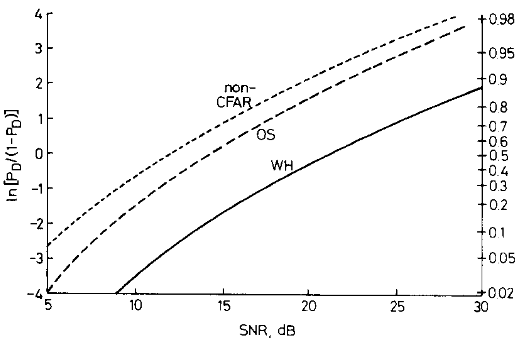


**Fig. 3** *Detection probability as a function of SNR*
——— two-parameter Weber–Haykin (WH) CFAR; – – – – single-parameter order statistics (OS) CFAR; ------- non-CFAR, fixed threshold detector; $M = 16$; $i = 3$; $j = 16$; $k = 12$; $P_{FA} = 10^{-5}$

Fig. 3 reveals a considerable CFAR loss in the Weber–Haykin (WH) algorithm. For the example given ($P_{FA} = 10^{-5}$, $M = 16$) the best WH-CFAR yields a loss of 9.4 dB (at $P_D = 0.5$) compared to only 2.3 dB loss when the best OS-CFAR is used. By choosing the best ranks ($i = 3$, $j = 16$) to reach the smallest loss in WH-CFAR, we give up the inherent immunity of OS-CFAR to the presence of interfering targets among the reference cells. As shown in Reference 6, when the representative rank is $k$, the OS-CFAR algorithm does not suffer a considerable additional loss as long as the number of interfering targets does not exceed $M - k$. To gain a similar immunity for the WH algorithm we must set $j < M$. However, it turns out that lowering $j$ adds considerable CFAR loss, as summarised in Table 1.

Recall that OS-CFAR with $k = 12$ suffers a CFAR loss of only 2.4 dB. If we would like the WH-CFAR to have immunity against the same number of interfering targets as OS-CFAR with $k = 12$, we will have to set $j = 12$. According to Table 1, this will raise the CFAR loss from 9.4 to 21.5 dB.

**Table 1: CFAR loss in decibels of the Weber–Haykin algorithm as function of the representative ranks *i, j***

| *i* | *j* | | | | | |
|---|---|---|---|---|---|---|
| | 16 | 15 | 14 | 13 | 12 | 11 |
| 3 | 9.4 | 11.0 | 13.5 | 17.0 | 21.5 | 27.5 |
| 4 | 9.9 | 12.0 | 14.9 | 18.9 | 24.5 | 32.7 |
| 5 | 10.8 | 13.2 | 17.0 | 22.0 | 29.3 | 41.2 |

$C = 2$; $P_{FA} = 10^{-5}$; $P_D = 0.5$; $M = 16$

For the special case of $C = 2$, the connection between the CFAR loss and the standard deviation of the estimated shape parameter, as found from eqn. 24, is shown in Fig. 4. Note that the data presented in Fig. 4 were calculated for different values of $M$, $i$ and $j$.

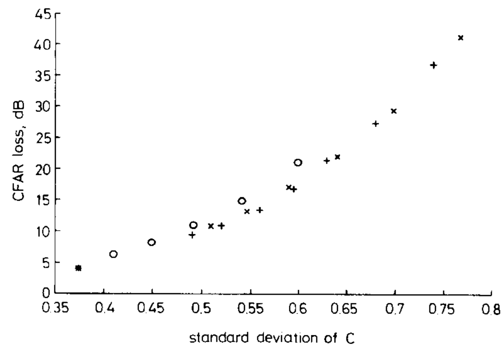


**Fig. 4** *CFAR loss of Weber–Haykin algorithm as a function of standard deviation of estimated shape parameter*
○ $M = 24$, $i = 5$; + $M = 16$, $i = 3$; × $M = 16$, $i = 5$; ✱ $M = 32$, $i = 5$, $j = 31$

## 5 Comparison with a two-parameter CFAR based on mean-variance technique

The two parameters of the Weibull distribution can also be estimated [7] using the calculated mean and variance of the log transformed variable ln $X$. A CFAR processor based on this principle was suggested by Hansen [3], and applied to a Weibull background. In his Fig. 15, Hansen provides CFAR loss results obtained through Monte Carlo simulations for a Rayleigh fluctuating target and for Weibull background with $C = 2$ (Rayleigh). Our theoretical analysis applies to this exact situation. We have therefore extended our analysis to $M = 32$ reference cells, which is the smallest number of reference cells used by Hansen. The optimal choice of representative ranks for $M = 32$ is $i = 5$ and $j = 31$. The CFAR loss as function of the $P_{FA}$, for $P_D = 0.5$, is shown in Fig. 5.

We reiterate that the WH-CFAR results are theoretical ones, based on using eqn. 32 to find $\beta$ and eqn. 38 to find $D$, which is related to the average $SNR$ by eqn. 34. Hansen's results were obtained by Monte Carlo simulations. The loss is obtained by comparison with a fixed threshold detector, which for a Rayleigh target in Rayleigh noise obeys the relationship

$$\overline{SNR} = \frac{\log P_{FA}}{\log P_D} - 1 \quad (39)$$

Fig. 5 shows that the detection loss of the WH-CFAR algorithm is smaller than the loss of the CFAR processor suggested by Hansen.

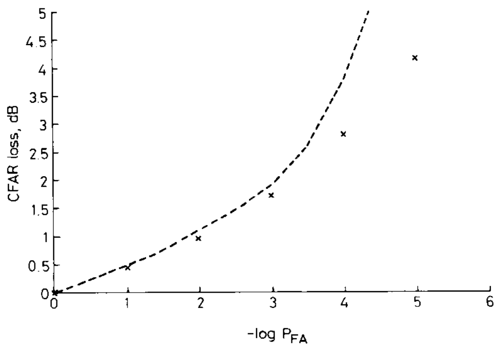


**Fig. 5** *CFAR loss of Weber–Haykin and Hansen CFAR algorithms*
- - - - Hansen; × Weber–Haykin, $i = 5, j = 31$; $M = 32$; $P_D = 0.5$

## 6 Threshold based on more than two ordered samples

Choosing $k = i$ to obtain eqn. 21 is only one of $M$ choices. A more general choice will result in a threshold based on three ranks with the format

$$T_X = X_j\left(\frac{X_k}{X_i}\right)^{\beta} \tag{40}$$

By rearranging the subscripts in eqn. 40 we can see some preference in the role of the different ordered samples. However, eqn. 40 need not be restricted to $i < j < k$.

The joint PDF of $X_i, X_j, X_k$ (when $X_i < X_j < X_k$) can be found [5] by generalising eqn. 27:

$$p_{ijk}(x, y, z) = \begin{cases} Q_3[P(x)]^{i-1}p(x)[P(y) - P(x)]^{j-i-1} \\ \times\, p(y)[P(z) - P(y)]^{k-j-1}p(z)[1 - P(z)]^{M-k} & x \leqslant y \leqslant z \\ 0 & \text{elsewhere} \end{cases} \tag{41}$$

where

$$Q_3 = \frac{M!}{(i-1)!(j-i-1)!(k-j-1)!(M-k)!} \tag{42}$$

The relation between $P_{FA}$ and $\beta$, when the threshold is set according to eqn. 40, is given by the triple integral

$$P_{FA} = \int_{z=0}^{\infty}\int_{y=0}^{z}\int_{x=0}^{y} \exp\left[-(x^{-\beta}yz^{\beta})^C\right] \times p_{ijk}(x, y, z)\, dx\, dy\, dz \tag{43}$$

Inserting the parent distributions of eqns. 8 and 26 and repeating the substitutions of eqn. 31 will remove the shape parameter $C$ from the relation, making it also a two parameter CFAR.

Numerical integration of eqn. 43 is rather tedious. Again an approximation to $\beta$ can be obtained from

$$\beta \simeq \beta^* = \frac{\ln \alpha_j}{\ln[-\ln(1 - h_k)] - \ln[-\ln(1 - h_i)]} \tag{44}$$

where $\alpha_j$ is found from eqn. 4.

To demonstrate this point we chose $i = 3$, $j = 10$, $k = 16$, $M = 16$ and $P_{FA} = 0.01$ in eqns. 4 and 44, which gave $\beta^* = 0.69064$. We then used $\beta^*$ instead of $\beta$ in eqn. 40 and performed Monte Carlo simulations on 40000 runs. The resultant value of $P_{FA}$ was 0.00988.

In the thresholds defined by eqns. 21 and 40 the shape parameter $C$ is estimated using two ranked samples. A better estimate could be reached by using more than two ranked samples. Dubey [4] suggested an estimator of $C$ based on all $M$ ranked samples as follows

$$\hat{C} = \frac{\sum_{i=1}^{j}\{\ln[-\ln(1 - h_{j+i})] - \ln[-\ln(1 - h_i)]\}}{\sum_{i=1}^{j}(\ln X_{j+i} - \ln X_i)} \tag{45}$$

where $2j = M$. Using this estimator in eqn. 17 will yield the threshold

$$T_X = X_k\left(\frac{X_{j+1}X_{j+2}\cdots X_M}{X_1X_2\cdots X_j}\right)^{\beta_M} \quad 2j = M \tag{46}$$

where

$$\beta_M = \frac{\ln \alpha_k}{\sum_{i=1}^{j}\{\ln[-\ln(1 - h_{j+i})] - \ln[-\ln(1 - h_i)]\}} \tag{47}$$

The exact relation between $\beta$ and the $P_{FA}$ will require an unwieldy $M$-tuple integral. That leaves eqn. 47, or Monte Carlo simulations, as the only practical means for obtaining $\beta$. Again we should point out that because of the variance in the estimate of $C$, the resulting $P_{FA}$ could be somewhat different from the $P_{FA}$ related to $\alpha_k$ through eqn. 4.

It is of interest to compare the performance of the various OS-CFAR algorithms discussed so far. A comparison was performed by calculating the average threshold obtained by Monte Carlo simulations of the various processors, when $\beta$ was adjusted to yield a $P_{FA} = 0.01$. The mean threshold $E\{T\}$ was proposed by Rohling [1] as a measure of the relative performances of CFAR processors. Clearly, a higher $E\{T\}$ will yield a higher detection loss. The results of the comparison, for the case of $C = 2$, are summarised in Table 2.

**Table 2: Monte Carlo results of CFAR processors**

| | 1 rank, $k = 12$ | 2 ranks, $i = 3, j = 16$ | 3 ranks, $i = 3, j = 10, k = 16$ | $M$ ranks, $k = 3$ | $M$ ranks, $k = 10$ |
|---|---|---|---|---|---|
| Eqn. | 17 | 21 | 40 | 46 | 46 |
| $\beta$ | — | 1.2333 | 0.69064 | 0.251 | 0.1423 |
| $P_{FA}$ | 0.01022 | 0.00973 | 0.01005 | 0.01015 | 0.00995 |
| $E\{T\}$ | 2.371 | 2.573 | 2.684 | 2.734 | 2.805 |
| $\sigma_T$ | 0.366 | 0.596 | 0.759 | 0.846 | 0.857 |
| $E\{C\}$ | — | 2.340 | 1.934 | 2.277 | 1.859 |
| $\sigma_c$ | — | 0.583 | 0.481 | 0.548 | 0.447 |

The last two rows present the mean value of the shape parameter, calculated in each trial using eqn. 19 or 45, and its standard deviation. They demonstrate that, whereas estimates of $C$ based on more ranks yield a smaller standard deviation of $C$, they do not produce a smaller mean threshold $E\{T\}$.

The simulations leading to Table 2 were performed with $M = 16$ reference cells. The first column gives, as a reference, the results obtained with the original OS-CFAR, using *a priori* knowledge of $C$. The second column gives Weber and Haykin's two-ranks OS-CFAR with an optimal choice of representative ranks. The remaining columns give processors using more than two ranks, whose thresholds are defined by the equations indicated in the first row.

The table demonstrates that, among the two parameter CFAR processors, the Weber–Haykin algorithm exhibits the lowest mean threshold and standard deviation and is, therefore, likely to exhibit the smallest CFAR loss. However, comparing $\sigma_C/E\{c\}$, reveals that an estimate of $C$ based on all the $M$ ranks (columns 4 and 5) yielded a smaller variance.

## 7 Should the shape parameter be estimated?

We have shown that the WH algorithm performs better than all the other two-parameter CFAR processors. However, even that algorithm resulted in a considerably higher loss than the single-parameter algorithm. For example, we learn from Table 1 that (with $M = 16$ reference cells and $P_{FA} = 10^{-5}$) the penalty for estimating the shape parameter is an additional loss of 7.1 dB. Such a considerable loss justifies asking the question whether the shape parameter should be estimated. The alternative is to set $\gamma_k$, using eqn. 18, to yield the required $P_{FA}$ at the lowest expected $C$. If the range of uncertainty in $C$ is not too large, the loss at the highest possible $C$ may still be smaller than when $C$ is estimated.

The exact CFAR loss can be calculated if $C_{max} = 2$ and the target is Rayleigh fluctuating. An example is demonstrated in Fig. 6. The short-dashed line represents

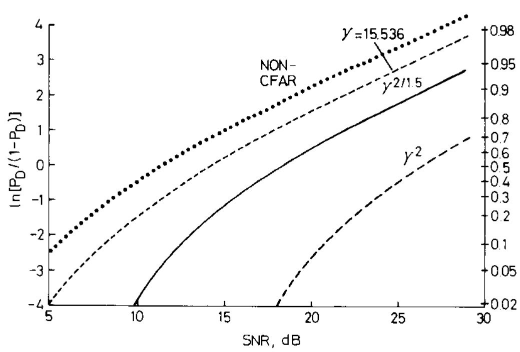


**Fig. 6** *Performance of OS-CFAR when $C = 2$, but threshold is set assuming $C_{min} \leqslant 2$*
······· $C_{min} = 2$; ——— $C_{min} = 1.5$; - - - - $C_{min} = 1$; $M = 16$, $k = 12$, $P_{FA} = 10^{-5}$

the performance of the regular OS-CFAR when $C = 2$, with no uncertainty. The scale factor was calculated from eqn. 18 as $\gamma_k = 15.536$. $P_D$ against $SNR$ was calculated using eqn. 18 after replacing $\gamma_k$ by $D\gamma_k$, where $D$ is as defined in eqn. 34.

The solid line represents the performances when $C_{min} = 1.5$. To limit $P_{FA}$ to $10^{-5}$ at that $C$, the scale factor should be $15.536^{2/1.5} = 38.766$. An additional loss is incurred when the resulting higher threshold operates with clutter whose actual shape parameter is $C = 2$. Comparing the solid line to the dashed line an additional loss of 4 dB can be seen. This loss is smaller than the 7.1 dB additional loss which would have occurred if $C$ had been estimated. The long-dashed line presents the performance when $C_{min} = 1$. In this case the additional loss (relative to a known $C$) is 12.2 dB.

We can conclude from Fig. 6 that, when the uncertainty in the shape parameter is $1.5 \leqslant C \leqslant 2$, it is better to assume $C = 1.5$ than to estimate $C$. However, if $1 \leqslant C \leqslant 2$, it is better to estimate $C$. In this latter case the conclusion ignores an important factor. In the regular OS-CFAR the highest rank used to derive the threshold was $k = 12$. This allows for $M - k = 4$ interfering targets to be present before the processing collapses [6]. On the other hand, in the WH-CFAR the highest rank utilised was $j = 16 = M$, which leaves no allowance for interfering targets. From Table 1 we can see that using a lower representative rank $j$ increases the loss drastically. For example, to allow for four interfering targets, $j$ should be 12, implying an additional loss (relative to a known $C$) of 19.2 dB. Thus, when interfering targets could be present, the use of the regular OS-CFAR should be recommended even for the large uncertainty range of $1 \leqslant C \leqslant 2$.

## 8 Discussion and conclusions

OS-CFAR is not the only CFAR algorithm that exhibits immunity to spurious targets. The censored mean-level detector [8] is an example of a CFAR technique that enjoys a similar immunity, though at a slightly smaller loss than OS-CFAR. OS-CFAR, however, has a unique feature in that the threshold is calculated from a *single* reference cell, selected from the $M$ reference cells by ranking. This feature allows for a simple adaptation of the technique to background distributions that are derived from Rayleigh by raising the random variable to any power. We have exploited this quality and have generalised Rohling's OS-CFAR threshold calculations to cover a Weibull clutter of arbitrary (but known) shape parameter. We have also provided a means of calculating the change in the false alarm probability when the shape parameter differs from the nominal one.

When the uncertainty in the shape parameter is too large, it is possible to resort to the Weber–Haykin two-parameter OS-CFAR algorithm. We have analysed the CFAR loss (for the special case of $C = 2$) of the algorithm and have compared it to that of another two-parameter CFAR processor suggested by Hansen. We found that the WH-CFAR exhibits a lower detection loss than Hansen's algorithm. In general, however, the need to estimate both parameters is very costly in terms of CFAR loss. We were able to derive analytic expressions for the detection performances only for the case of $C = 2$ and therefore our results regarding detection losses are limited to that case; we feel, however, that they represent a broad range of the shape parameter $C$. Finally, we found that modifications of the Weber–Haykin algorithm, in which the threshold is set by more than two ranks, yield even poorer results.

## 9 References


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