Cross-correlation of long binary signals with longer mismatched filters

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Abstract: Mismatched processing of long binary signals is revisited. The filter is optimised for minimum integrated or peak sidelobes. The importance of choosing a signal with favourable autocorrelation is demonstrated using a few examples.

1 Introduction

Binary signals are relatively easy to generate in radar transmitters. In order to achieve large pulse compression ratios, there is a continuous search for longer and longer binary signals whose aperiodic autocorrelation function exhibits low peak sidelobes [1, 2] or low integrated sidelobes [3–6]. If some SNR loss is acceptable, then it is possible to use a mismatched filter, which, when correlated with the signal, yields cross-correlation output with lower peak (or integrated) sidelobes, without too much SNR loss. Mismatched filters can be optimised for polyphase codes as well as for binary codes. For a given signal, the optimisation is straightforward when the criterion is minimum integrated sidelobes, and more demanding when the criterion is minimum peak sidelobe [7–11].

In an extended recent study Nunn [12] suggests that the signal/filter optimisation should start from a signal whose own autocorrelation function already exhibits low peak or integrated sidelobes. This observation is examined here using relatively long binary signals. The mismatched filters are three times longer than the signal.

Section 2 defines the sequence and the performance criteria. Section 3 gives examples of ISL (integrated sidelobes) and PSL (peak sidelobe) filters optimised for the Barker 13 signal. Section 4 shows performances of two different signals of length 63. Section 5 compares the performances of three different signals of length 169. Section 6 shows the dependence on filter length.

2 Definitions

The binary sequence is given by

$$S = \{s_1 = \pm 1, s_2, \ldots, s_N\} \tag{1}$$

2.1 Matched filter

The output of the matched filter (without Doppler shift) is the aperiodic autocorrelation function, whose values for positive delays are given by

$$C_k(S) = \sum_{i=1}^{N-k} s_i\, s_{i+k}, \quad k = 0, 1, 2, \ldots, N-1 \tag{2}$$

Because the signal is real valued, the autocorrelation is real and symmetric about the zero delay. The energy in the autocorrelation sidelobes (positive delays) is

$$E(S) = \sum_{k-1}^{N-1} C_k^2(S) \tag{3}$$

Because of the symmetry of the autocorrelation, the total sidelobe energy is $2E(S)$. The merit factor, which corresponds to the inverse of the normalised integrated sidelobes, is

$$F = \frac{N^2}{2E(S)} \tag{4}$$

Barker code of length 13 has the largest merit factor (=14.083). A typical value of F for good very long codes is 7.

2.2 Mismatched filter

The filter elements are

$$H = \{h_1, h_2, \ldots, h_M\} \tag{5}$$

where the elements are real and $N \le M$. For simplicity we will assume that if N is odd then M is also odd, and when N is even M is also even. This implies that $M - N$ is always even, hence $(M-N)/2 = z$ is an integer. We will now define Z as an all-zero sequence of length z, and create a zero-padded signal sequence of length $M = N + 2z$ given by

$$S_0 = \{Z\ S\ Z\} \tag{6}$$

Clearly, the sequences H and S_0 are both of equal length M. We will also assume that the filter is designed so that the cross-correlation $R_k(H, S_0)$ between H and S_0 will peak at zero delay ($k = 0$). $R_k(H, S_0)$ is not necessarily symmetric around zero delay.

The integrated sidelobe energy ratio will be defined as

$$ISLR = \frac{1}{R_0^2} \sum_{k \ne 0} R_k^2 \tag{7}$$

When the filter is matched, $R_0^2 = C_0^2(S) = N^2$ and (4) and (7) imply that $ISLR = 1/F$. The elements of the mismatched

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filter are normalised to yield the same noise output power as the matched filter, when the input is only white noise. Thus the normalisation requires that

$$HH^T = SS^T \tag{8}$$

where the superscript $(\cdot)^T$ implies transpose operation. With that normalisation, the SNR loss due to the mismatched filter becomes

$$L = \frac{R_0^2}{C_0^2(S)} = \frac{R_0^2}{N^2} \tag{9}$$

Another possible definition for the mismatched output integrated sidelobe ratio is

$$ISLR_2 = \frac{1}{C_0^2(S)} \sum_{k \neq 0} R_k^2 = \frac{1}{N^2} \sum_{k \neq 0} R_k^2 = ISLR \cdot L \tag{10}$$

In the $ISLR_2$ definition, the energy in the sidelobes at the output of the normalised mismatched filter is compared to the peak output of the matched filter.

The other important ratio is the peak sidelobe ratio, *PSLR*. Here, again, two possible definitions exist. In the first, the highest cross-correlation sidelobe is compared to the peak cross-correlation output:

$$PSLR = \frac{1}{R_0^2} \left(\max_{k \neq 0} |R_k| \right)^2 \tag{11}$$

In the second definition, the peak cross-correlation sidelobe is compared to the peak autocorrelation, namely

$$PSLR_2 = \frac{1}{C_0^2(S)} \left(\max_{k \neq 0} |R_k| \right)^2 = \frac{1}{N^2} \left(\max_{k \neq 0} |R_k| \right)^2 = PSLR \cdot L \tag{12}$$

3 Barker 13

The well-known matched filter response of Barker 13 has an *ISLR* of 0.071 $(= 1/14.083)$, or -11.487 dB. The *PSLR* is 0.0059 $(= 1/13^2)$, or -22.289 dB. The mismatched filter's length will be three times the signal length, namely $M = 3N = 39$. The resulted absolute values of the cross-correlations are plotted in Figs. 1 and 2. The ISL optimised filter was obtained using a conventional least-squares

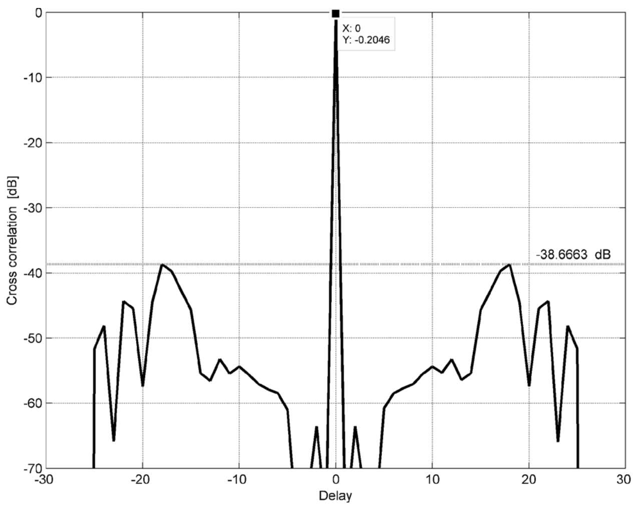

Fig. 1 *Output of mismatched min ISL filter with $M = 39$, for a Barker 13 signal*

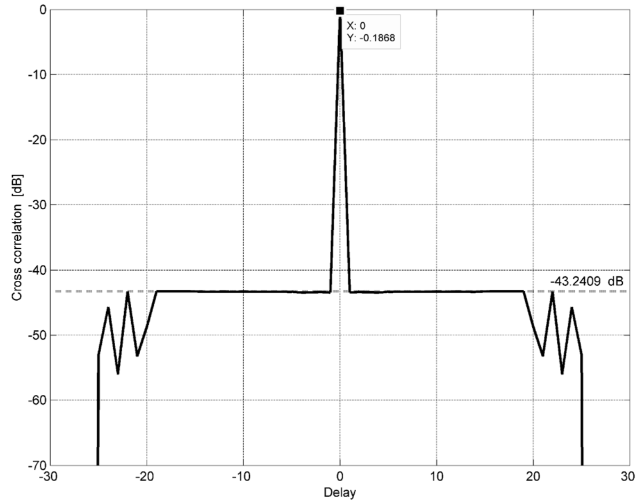

Fig. 2 *Output of mismatched min PSL filter with $M = 39$, for a Barker 13 signal*

approach [11]. The PSL optimised filter was obtained using MATLAB's constrained optimisation function `fmincon`. Note that in both drawings the 0 dB level corresponds to the peak autocorrelation value. Thus, Fig. 1 indicates that the mismatched ISL optimised filter yields an SNR loss of 0.2046 dB and *ISLR* of -30.03 dB. Figure 2 indicates that the mismatched PSL optimised filter exhibits an SNR loss of 0.1868 dB, and the $PSLR_2$ is -43.241 dB.

The 13 elements of a filter matched to a Barker 13 signal S receive the same ± 1 values of the signal. The 39 elements of a mismatched filter clearly deviate from the 0 and ± 1 values of the zero-padded signal S_0. The deviation is plotted in Fig. 3. It is interesting to note from Fig. 3 that the elements of the min ISL and min PSL mismatched filters are not very different from each other. Yet the responses are quite different. The small differences between corresponding elements of the two filters (typical 0.02 compared to matched filter values of ± 1) should hint about what is an acceptable quantisation of the mismatched filter elements.

4 Binary signals of length $N = 63$

In this Section we will use two different binary signals, as an example of the importance of using a signal whose

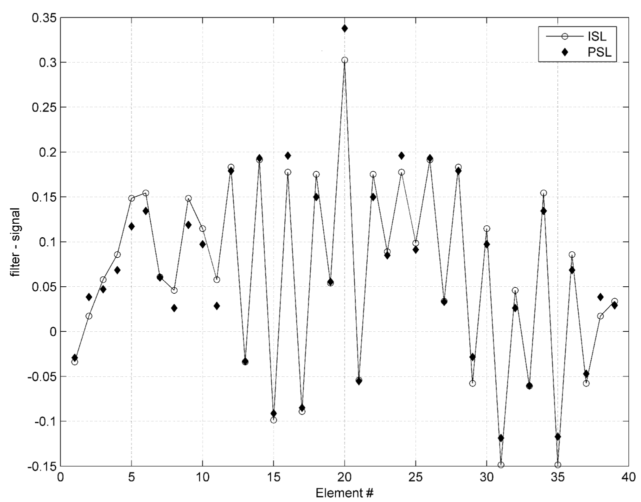

Fig. 3 *Deviation of mismatched filter elements from Barker 13 signal values (the signal occupies elements 14 to 26)*

initial matched response is good. The two signals are the minimum PSL signal of length 63, which exhibits a matched *PSLR* of $(4/63)^2$ or -23.946 dB, and an m-sequence of the same length, whose matched *PSLR* is $(6/63)^2$ or -20.424 dB.

A min PSL signal of length 63 appears in [1]; its peak sidelobe is 4. The signal is

\+ + + + − + + − − +   + − − − − + − + − −
\+ − + + + − + + − +   + + − − − + + + − +
− − + + − + − + − −   − − − + + + − − + −
− − −

Of the several m-sequences of length 63, the lowest attainable peak sidelobe is 6. An example of such an m-sequence is

\+ + + + + − + − + −   + + − − + + − + + +
− + + − + − − + − −   + + + − − − + − + +
\+ + − − + − + − − −   + + − − − − + − − −
− − +

The responses of a mismatched filter of length 189, optimised for minimum PSL, to the min PSL signal of length 63 are plotted in Fig. 4, and to the m-sequence signal in Fig. 5. Comparing the two plots we note a very similar SNR loss of about 1.26 dB. However, the $PSLR_2$ are -39.145 dB and -35.747 dB, respectively. Namely, the advantage of 3.5 dB in the matched response of the min PSL signal over the m-sequence signal was maintained in the mismatched response.

5 Binary signals of length $N = 169$

In this Section we will use three different binary signals, and compare their *ISLR* and *PSLR* at the output of the matched filters with the outputs of optimised min ISL and min PSL mismatched filters of length 507.

The first signal was described in [6]; it exhibits a merit factor of 9.321, which corresponds to matched filter *ISLR*

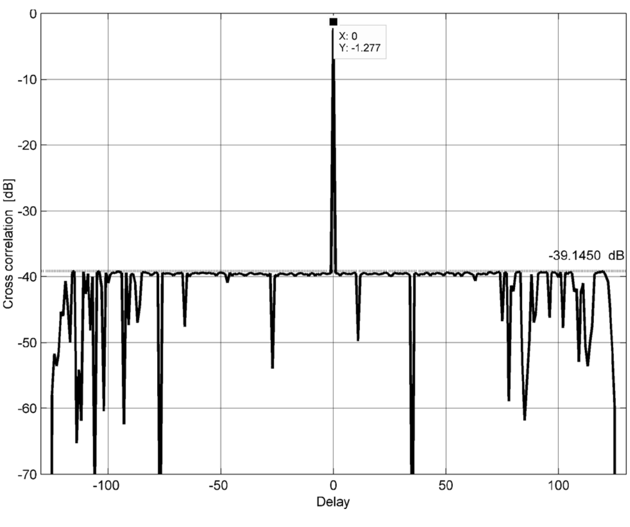

Fig. 4 *Output of mismatched min PSL filter with $M = 189$, to a min PSL signal of length 63*

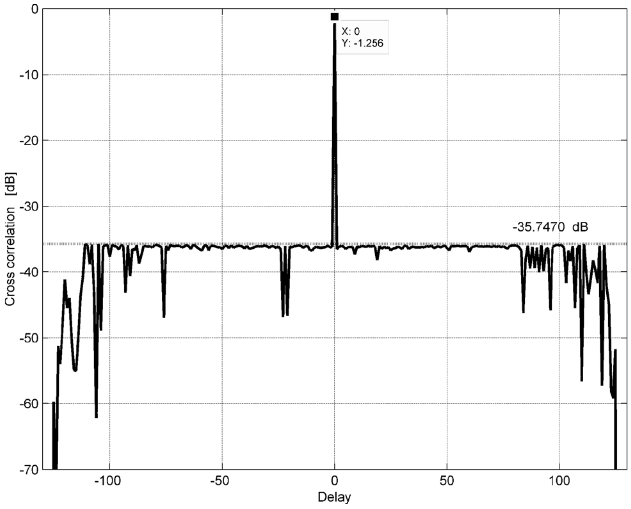

Fig. 5 *Output of mismatched min PSL filter with $M = 189$, to an m-sequence signal of length 63*

of -9.695 dB.

\+ + + − − − − − + + + − − − + + + + + − − − + +
− − − + − − − + + + − + + + − − − + + + − + −
\+ + − − − + + + + + + + − + − − − − − + − + +
\+ − + + − + − + + + + + + − + + + − + − + − −
− − − + + + − + + + + − + − + + + + − + − + −
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\+ − − − + − − + + − + + − + − + + − + + − + +
− + − + + − +

An effective format for presenting long binary codes is the run-length format, which for the above code is given by:

3533532313313331112371151131211161311115314111 4
111112125121213121312212111212121112 11

The second signal is a Barker 13 nested in a Barker 13, which yields the signal

1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
−1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1	−1
−1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1	−1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
−1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1	−1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1
−1	−1	−1	−1	−1	1	1	−1	−1	1	−1	1	−1
1	1	1	1	1	−1	−1	1	1	−1	1	−1	1

The third signal is a chaotic signal [13], generated using the logistic-map equation

$$x_{n+1} = r x_n (1 - x_n) \tag{13}$$

with $r = 4$, $x_1 = 0.1$, and $n = 1$ to 169. The conversion to a binary signal follows

$$s_n = \begin{cases} -1, & x_n \leq 0.5 \\ +1, & x_n > 0.5 \end{cases} \tag{14}$$

yielding the signal

$$\begin{array}{rrrrrrrrrrrrr}
-1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 \\
-1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\
1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
-1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1
\end{array}$$

Figures 6 to 8 present the cross-correlations of each signal with its mismatched filter (of length 507) optimised for minimum ISL. The performances are listed in Table 1. Figures 9–12 present the cross-correlations of each signal with its mismatched filter (of length 507) optimised for minimum PSL. The performances are listed in Table 2. In contrast to the other two signals, in the case of the chaotic signal, the SNR loss was the determining constraint.

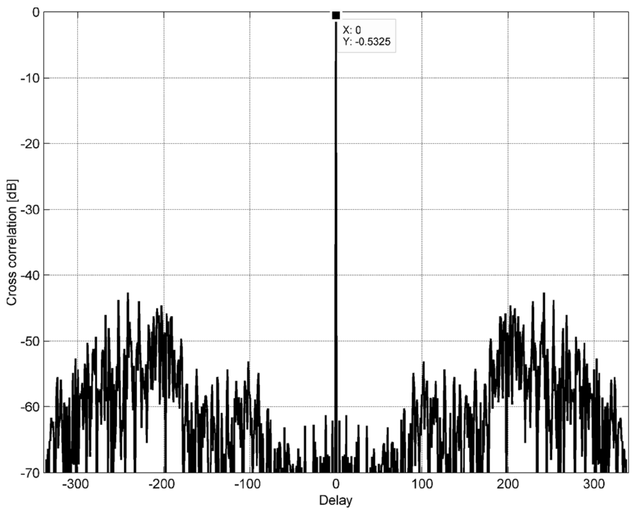

Fig. 6 *Output of mismatched min ISL filter with M = 507, to a low ISL signal of length 169*

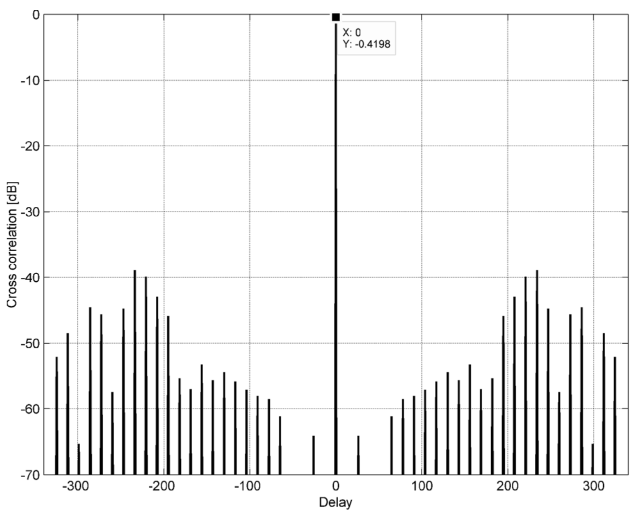

Fig. 7 *Output of mismatched min ISL filter with M = 507, to a 13 × 13 nested Barker signal*

As shown in Table 2 and Figs. 11 and 12, the SNR loss constraint (−1.63 or −5.47 dB) determined the min PSL filter and the $PSLR_2$ level. In the other two signals the SNR loss constraint was −1.63 dB, and it was not reached.

The first two rows of Table 1 show a slight deviation from the assumption that a better matched filter *ISLR* leads to better mismatched filter *ISLR*. However, even when the filter is optimised for min ISL, the resulting $PSLR_2$ is also of concern. Comparing Figs. 6 and 7 shows that whereas Fig. 7 yields better *ISLR* (see Table 1), the $PSLR_2$ is considerably worse (by approximately 4 dB) than in Fig. 6.

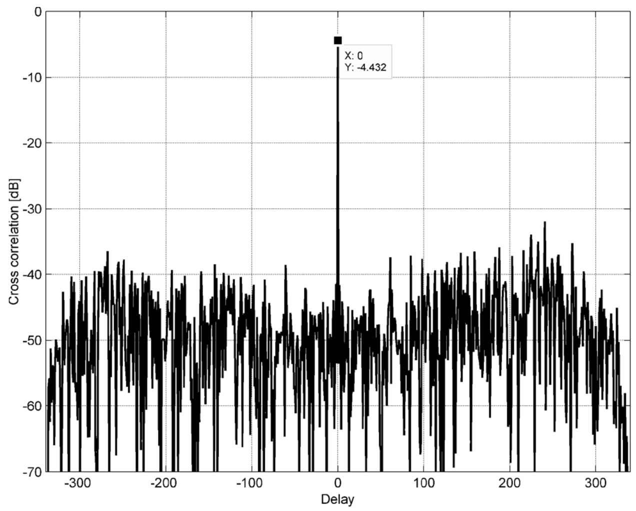

Fig. 8 *Output of mismatched min ISL filter with M = 507, to a chaotic signal of length 169*

Table 1: Performances of mismatched filters of length 507, optimised for minimum ISL, to three binary signals of length 169

Signal	*ISLR* matched, dB	*ISLR* mismatched, dB	SNR loss, dB
Low ISL	−9.69	−26.69	−0.53
Nested Barker 13 × 13	−8.33	−30.02	−0.42
Chaotic	−1.90	−12.44	−4.43

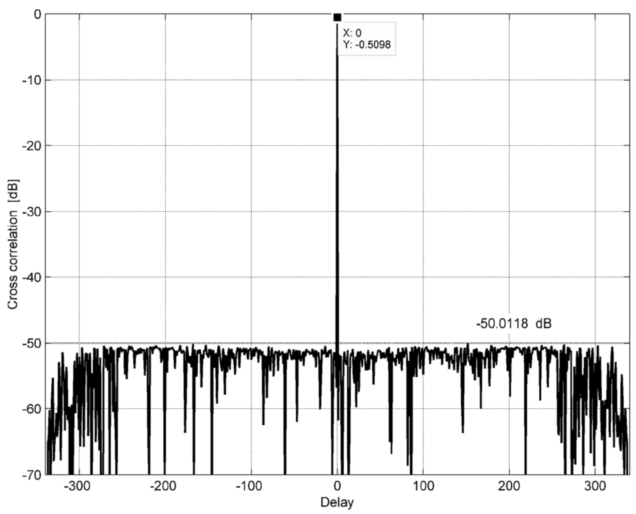

Fig. 9 *Output of mismatched min PSL filter with M = 507, to a low ISL signal of length 169*

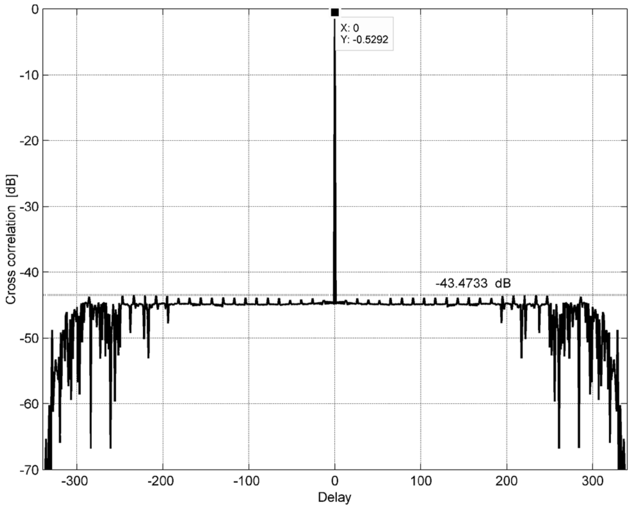

Fig. 10 *Output of mismatched min PSL filter with $M = 507$, to a nested 13×13 Barker signal of length 169*

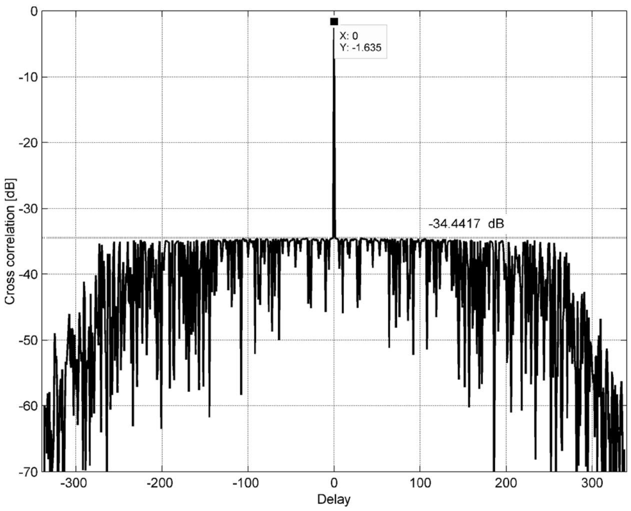

Fig. 11 *Output of mismatched min PSL filter with $M = 507$, to a chaotic signal of length 169, with the SNR loss constrained to -1.635 dB*

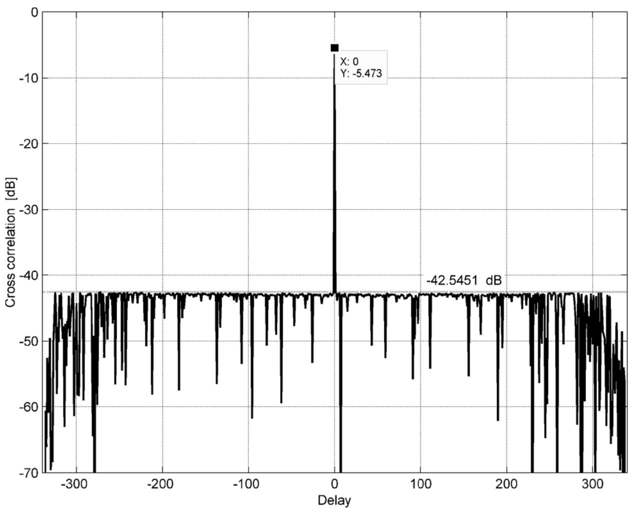

Fig. 12 *Output of mismatched min PSL filter with $M = 507$, to a chaotic signal of length 169, with the SNR loss constrained to -5.473 dB*

Table 2: Performances of mismatched filters of length 507, optimised for minimum PSL, to three binary signals of length 169

Signal	*PSLR* matched, dB	$PSLR_2$ mismatched, dB	SNR loss, dB
Low ISL	−23.73	−50.01	−0.51
Nested Barker 13 × 13	−22.28	−43.47	−0.53
Chaotic	−18.54	−34.44	−1.63
		−42.55	−5.47

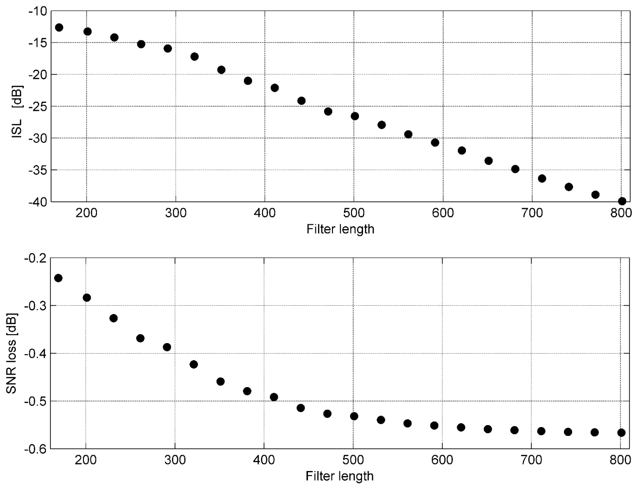

Fig. 13 *ISLR and SNR loss dependence on the mismatched filter length, for a 169-element binary signal*

6 *ISLR* dependence on filter length

So far, we have used a single filter length, three times the length of the signal. Fig. 13 demonstrates the dependence of *ISLR* and SNR loss on the length of a mismatched filter, optimised for minimum ISL, when the signal is the first of the three signals of length 169. We see the *ISLR* continuously dropping as the filter length grows, while the SNR loss levels at about −0.57 dB.

7 Conclusions

Several examples help strengthen the intuitively appealing assumption that mismatched filter response, optimised for minimum ISL or PSL, produces better pulse compression performances when the signal matched response is initially favourable. The examples included long binary signals and mismatched filters three times as long.

8 Acknowledgment

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