third harmonic has also been computed and found to vary from 9.1 to 1.4 percent when dc bias is zero and 0.2 V, respectively. The second harmonic thus predominates over the whole range and is more important at lower bias voltages.

References

[1] M. S. Shur and L. F. Eastman, "Ballistic transport in semiconductor at low temperature for low-power high-speed logic," *IEEE Trans. Electron Devices*, vol. ED-26, no. 11, Nov. 1979.

[2] L. F. Eastman, R. Stall, D. Woodland, N. Dandekar, C.E.C. Wood, M. S. Shur, and K. Board, "Ballistic electron transport in GaAs at room temperature," *Electron. Lett.*, vol. 16, no. 13, 1980.

Convergence of Polynomial Least Squares End-Point and Mid-Point Estimators

NADAV LEVANON

***Abstract*—In a recent correspondence, Leskiw and Miller [1] obtained the variance of a least squares polynomial estimator, as function of the polynomial order, for a large number of equally spaced data points, when the estimate is for the end-point. A simpler proof is given, and the result is extended to the mid-point. A mid-point estimator may be of special interest since it exhibits the lowest variance.**

Following the analysis in [1], let

$$x(t) = a_p + a_{p-1}t + \cdots + a_1 t^{p-1} + a_0 t^p \tag{1}$$

be a polynomial of degree p where the coefficients $a_k, k = 0, 1, \cdots, p$ are unknown. Suppose we are given n equally spaced observations z_k, on x where

$$z_k = x(k\Delta t) + u_k, \quad k = 1, 2, \cdots, n \tag{2}$$

and u_k is an additive noise. In matrix notation we may write

$$\boldsymbol{z} = H_p \boldsymbol{a} + \boldsymbol{u} \tag{3}$$

where $\boldsymbol{z}' = \{z_1, \cdots, z_n\}$ and $\boldsymbol{u}' = \{u_1, \cdots, u_n\}$ are n-dimensional row vectors and $\boldsymbol{a}' = \{a_p, \cdots, a_0\}$ is a $(p + 1)$-dimensional row vector (prime denotes transpose). The $n \times (p + 1)$ matrix H_p in (3) is given by

$$H_p = \begin{bmatrix} 1 & \Delta t & (\Delta t)^2 & \cdots & (\Delta t)^p \\ 1 & 2\Delta t & (2\Delta t)^2 & \cdots & (2\Delta t)^p \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & n\Delta t & (n\Delta t)^2 & \cdots & (n\Delta t)^p \end{bmatrix}. \tag{4}$$

We assume that the noise has a zero mean and that its covariance matrix is given by

$$\text{Cov}\, \boldsymbol{u} = \sigma^2 I \tag{5}$$

where I is an $n \times n$ identity matrix.

At this point we depart from the analysis of [1] and note that due to symmetry the variances of the estimates at the two end-points should be equal, i.e.,

$$\text{Var}\, x(0) = \text{Var}\, x(n\Delta t). \tag{6}$$

We also note that

$$\text{Var}\, x(0) = \text{Var}\, a_p \tag{7}$$

and that is well known to be given by

$$\text{Var}\, a_p = \sigma^2 \{A_p^{-1}\}_{1,1} \tag{8}$$

where

$$A_p = H_p' H_p. \tag{9}$$

We will now obtain a simple expression for A_p, using the fact that the number of observations $-n$ is large [6].

Using (4) in (9) and multiplying and dividing by Δt we get

$$A_p = \frac{1}{\Delta t}\begin{bmatrix} \sum_{i=1}^{n} \Delta t & \Sigma(i\Delta t)\Delta t & \cdots & \Sigma(i\Delta t)^p \Delta t \\ \Sigma(i\Delta t)\Delta t & \Sigma(i\Delta t)^2 \Delta t & \cdots & \Sigma(i\Delta t)^{p+1}\Delta t \\ \vdots & \vdots & & \vdots \\ \Sigma(i\Delta t)^p \Delta t & \Sigma(i\Delta t)^{p+1}\Delta t & \cdots & \Sigma(i\Delta t)^{2p}\Delta t \end{bmatrix}. \tag{10}$$

Now let us define

$$i\Delta t = t \tag{11}$$

and

$$n\Delta t = T \tag{12}$$

where T is the total observation period. Finite T and large n imply $\Delta t << T$. Hence it is permissible to replace the sum by an integral. Thus

$$\sum_{i=1}^{n} (i\Delta t)^k \Delta t \approx \int_{t=0}^{T} t^k \, dt = \frac{T^{k+1}}{k+1}. \tag{13}$$

Using (12) and (13) in (10) we get

$$A_p = n \begin{bmatrix} 1 & \frac{T}{2} & \cdots & \frac{T^p}{p+1} \\ \frac{T}{2} & \frac{T^2}{3} & \cdots & \frac{T^{p+1}}{p+2} \\ \vdots & \vdots & & \vdots \\ \frac{T^p}{p+1} & \frac{T^{p+1}}{p+2} & \cdots & \frac{T^{2p}}{2p+1} \end{bmatrix}. \tag{14}$$

As was pointed out in [1], $\{A_p^{-1}\}_{1,1}$ is simply given by

$$\{A_p^{-1}\}_{1,1} = \frac{(p+1)^2}{n} \tag{15}$$

which leads to the result obtained in [1].

$$\text{Var}\, x(0) = \text{Var}\, x(T) = \sigma^2 \frac{(p+1)^2}{n}. \tag{16}$$

We will now extend this result to the mid-point and seek Var $x(T/2)$. This is easily done by shifting $t = 0$ to the mid-point and noting, similarly to (13), that

$$\sum_{i=-n/2}^{n/2} (i\Delta t)^k \Delta t \approx \int_{t=-T/2}^{T/2} t^k \, dt = \begin{cases} \dfrac{T^{k+1}}{2^k(k+1)}, & k+1 \text{ odd} \\ 0, & k+1 \text{ even} \end{cases}. \tag{17}$$

Using (12) and (17) in (10) we get

$$A_p = n \begin{bmatrix} 1 & 0 & \frac{(T/2)^2}{3} & 0 & \cdots & \frac{(T/2)^p}{p+1} \\ 0 & \frac{(T/2)^2}{3} & 0 & \frac{(T/2)^4}{5} & \cdots & 0 \\ \frac{(T/2)^2}{3} & 0 & \frac{(T/2)^4}{5} & 0 & \cdots & \frac{(T/2)^{p+2}}{p+3} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ \frac{(T/2)^p}{p+1} & \cdots & \cdots & \cdots & \cdots & \frac{(T/2)^{2p}}{2p+1} \end{bmatrix}. \tag{18}$$

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The format of (18), as written, assumes that p is even. For odd p, the last column and row start with 0.

It can be shown [2, pp. 429–431] that

$$\{A_p^{-1}\}_{1,1} = \frac{1}{n}(r!!)^2 \left[\left(\frac{r-1}{2}\right)!\right]^{-2} 2^{-(r-1)} \tag{19}$$

where

$$r = \begin{cases} p, & p \text{ odd} \\ p+1, & p \text{ even} \end{cases}$$

and where

$$r!! = 1 \times 3 \times 5 \times \cdots \times r. \tag{20}$$

Using an expression for an odd double factorial in (19), and (19) in (8), we finally get

$$\operatorname{Var} x(T/2) = \frac{\sigma^2}{n}\left\{r!\, 2^{-(r-1)}\left[\left(\frac{r-1}{2}\right)!\right]^{-2}\right\}^2 \tag{21}$$

where r is as in (19). Note that for large r, (21) converges to

$$\operatorname{Var} x(T/2) \approx \frac{\sigma^2}{n}\frac{2}{\pi} r, \qquad r \gg 1. \tag{22}$$

The following table summarizes the end-point and mid-point variances for $p \leqslant 5$

P	0	1	2	3	4	5
$\frac{n}{\sigma^2}\operatorname{Var} x(T/2)$	1	1	$\frac{9}{4}$	$\frac{9}{4}$	$\frac{225}{64}$	$\frac{225}{64}$
$\frac{n}{\sigma^2}\operatorname{Var} x(T)$	1	4	9	16	25	36

The table clearly shows the advantage of using the mid-point polynomial least squares estimator, over the end-point one, particularly when a high polynomial order is used.

Explicit expressions for the end-point and mid-point variances, not limited to a large n, are given in [3], for first- and second-degree polynomials. These expressions converge, for large n, to the results which appear in the table.

Comment added on January 6, 1983

In a recent comment to [1], Terzian [4] points out that Proschan [5] has given a general expression for the variance of any derivative at the end-point of the observation. Following Proschan's approach we were able to obtain a general expression for the variance of any derivative at the mid-point. Thus

$$\operatorname{Var} x^{(q)}(T/2) = \frac{\sigma^2}{n}\left(\frac{2}{T}\right)^{2q}\left[\frac{(s+q+1)!}{2^s\left(\frac{s+q+1}{2}\right)!\left(\frac{s-q-1}{2}\right)!}\right]^2 \frac{1}{2q+1} \tag{23}$$

where

$$s = \begin{cases} p, & \text{when } p+q = \text{odd} \\ p+1, & \text{when } p+q = \text{even}. \end{cases}$$

Recall that q is the order of the derivative, p is the polynomial order, n is the number of measurements, and T is the total observation time.

References

[1] D. M. Leskiw and K. S. Miller, "Convergence of polynomial least squares estimators," *Proc. IEEE*, vol. 70, pp. 520–522, 1982.
[2] T. Muir, *A Treatise on the Theory of Determinants.* New York: Dover, 1960.
[3] P. Z. Peebles, Jr., "An alternate approach to the prediction of polynomial signals in noise from discrete data," *IEEE Trans. on Aerosp. Electron. Syst.*, vol. AES-6, pp. 534–543, 1970.
[4] R. C. Terzian, "Comments on 'Convergence of polynomial least squares estimators'," *Proc. IEEE*, vol. 70, p. 1459, Dec. 1982.
[5] F. Proschan, "Precision of least-squares polynomials," *SIAM Rev.*, vol. 3, pp. 230–236, 1961.
[6] E. Weinstein, "Estimation of trajectory parameters from passive array processing," Ph.D. dissertation, Yale University, New Haven, CT, 1978.

Load Frequency Sampled-Data Control with Unknown Deterministic Power Demand

GORO SHIRAI

***Abstract*—This letter presents a sampled-data load frequency control method using the Lyapunov function. The proposed method is designed so as to absorb the unknown deterministic power demand. A numerical illustration is presented in order to verify the practicality of the proposed method.**

Introduction

Since the incremental power demand in a power system is not always *a priori* known, direct application of the modern control theory to the load frequency control (LFC) is not possible. In order to solve this problem, two main methods have been proposed [1], [2]. One is to identify the random power demands using the observer theory. The other is to introduce the propotional-plus-integral type optimal control strategy to absorb the load disturbances. In this letter, the identification of the unknown power demand is carried out by using the pseudoinverse matrix. The objective of this letter is to develop a simple and easily implematable sampled-data LFC strategy which can include the unknown deterministic power demand. The background theory of the proposed method is based on the Lyapunov function.

Formulation

Consider a linear time-invariant dynamic system described by

$$\dot{x} = Ax + Bu + Fz \tag{1}$$

where $x \in R^n$ is the state of the system, $u \in R^m$ is the control, $z \in R^r$ is the disturbance, and A, B, and F are real constant matrices of appropriate size. Constraints are imposed on the control variables

$$|u_i| \leqq c_i < \infty, \qquad i = 1, 2, \cdots, m. \tag{2}$$

In order to synthesize a sampled-data controller, let us transform (1) into the discrete-time equation represented by

$$x(t_{k+1}) = \Phi(T)\,x(t_k) + \Delta(T)\,u(t_k) + \Lambda(T)\,z(t_k) \tag{3}$$

where it was assumed that $t_{k+1} - t_k = T$ = constant for all k, and

$$\begin{aligned} \Phi(T) &= \exp(AT) \\ \Delta(T) &= [\exp(AT) - I]\,A^{-1}B \\ \Lambda(T) &= [\exp(AT) - I]\,A^{-1}F \end{aligned} \tag{4}$$

where I is an identity matrix.

In order to use the Lyapunov function, we redefine the states in terms of steady-state variables $x_e = x(t_k \to \infty)$, $u_e = u(t_k \to \infty)$ and $z_e = z(t_k \to \infty)$, i.e.,

$$\begin{aligned} X(t_k) &= x(t_k) - x_e \\ U(t_k) &= u(t_k) - u_e \\ Z(t_k) &= z(t_k) - z_e. \end{aligned} \tag{5}$$

Substituting (5) into (3), and employing $t_k \to \infty$ in (3), we have

$$X(t_{k+1}) = \Phi(T)\,X(t_k) + \Delta(T)\,U(t_k) + \Lambda(T)\,Z(t_k). \tag{6}$$

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